ELEMENTS OF ALGEBRA:

ON

THE BASIS OF M. BOURDON:

EMBRACING

STURM'S AND HORNER'S THEOREMS,

AND

PRACTICAL EXAMPLES.

BY CHARLES DAVIES, LL.D.

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NEW YORK:
A. S. BARNES & CO., 111 & 113 WILLIAM STREET,
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1868.

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PREFACE

The Treatise on Algebra, by M. Bourdon, is a work of singular excellence and merit. In France, it has long been one of the standard Text books. Shortly after its first publication, it passed through several editions, and has formed the basis of every subsequent work on the subject of Algebra, both in Europe and in this country.

The original work is, however, a full and complete treatise on the subject of Algebra, the later editions containing about eight hundred pages octavo. The time which is given to the study of Algebra, in this country, even in those seminaries where the course of mathematics is the fullest, is too short to accomplish so voluminous a work, and hence it has been found necessary either to modify it essentially, or to abandon it altogether.

In the following work, the original Treatise of Bourdon has been regarded only as a model. The order of arrangement, in many parts, has been changed; new rules and new methods have been introduced: the modifications indicated by its use, for twenty years, as a text book

in the Military Academy have been freely made, for the purpose of giving to the work a more practical character, and bringing it into closer harmony with the trains of thought and improved systems of instruction which prevail in that institution.

But the work, in its present form, is greatly indebted to the labors of William G. Peck, A. M., U. S. Topographical Engineers, and Assistant Professor of Mathematics in the Military Academy.

Many of the new definitions, new rules and improved methods of illustration, are his. His experience as a teacher of mathematics has enabled him to bestow upon the work much valuable labor which will be found to bear the marks of profound study and the freshness of daily instruction.

FISHKILL LANDING,
May, 1858.

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INTRODUCTION.

Quantity is a general term applicable to everything which can be increased or diminished, and measured. There are two kinds of quantity;

1st. Abstract quantity, or quantity, the conception of which does not involve the idea of matter; and,

2dly. Concrete quantity, which embraces every thing that is material.

Mathematics is the science of quantity; that is, the science which treats of the measurement of quantities, and of their relations to each other. It is divided into two parts:

1st. The Pure Mathematics, embracing the principles of the science and all explanations of the processes by which these principles are derived from the abstract quantities, Number and Space: and,

2d. The Mixed Mathematics, embracing the applications of these principles to all investigations involving the laws of matter, to the discussion of all questions of a practical nature, and to the solution of all problems, whether they relate to abstract or concrete quantity.*

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* Davies' Logic and Utility of Mathematics. Book II.

There are three operations of the mind which are imme diately concerned in the investigations of mathematical science: 1st. Apprehension; 2d. Judgment; 3d. Reasoning.

1st. Apprehension is the notion, or conception of an idea in the mind, analogous to the perception by the senses.

2d. Judgment is the comparing together, in the mind, two of the ideas which are the objects of Apprehension, and pro nouncing that they agree or disagree with each other. Judg ment, therefore, is either affirmative or negative.

3d. Reasoning is the act of proceeding from one judgment to another, or of deducing unknown truths from principles already known. Language affords the *signs* by which these operations of the mind are expressed and communicated. An *apprehension*, expressed in language, is called a *term;* a *judgment*, expressed in language, is called a *proposition;* and a *process of reasoning*, expressed in language, is called a *demonstration*.*

The reasoning processes, in Logic, are conducted usually by means of words, and in all complicated cases, can take place in no other way. The words employed are *signs of ideas*, and are also one of the principal instruments or helps of thought; and any imperfection in the instrument, or in the mode of using it, will destroy all ground of confidence in the result. So, in the science of mathematics, the meaning of the terms employed are accurately defined, while the language arising from the use of the symbols, in each branch, has a definite and precise signification.

* Whately's Logic,—of the operations of the mind and senses.

In the science of numbers, the ten characters, called figures, are the alphabet of the arithmetical language; the combinations of these characters constitute the pure language of arithmetic; and the principles of numbers which are unfolded by means of this, in connection with our common language, constitute the science.

In Geometry, the signs which are employed to indicate the boundaries and forms of portions of space, are simply the straight line and the curve; and these, in connection with our common language, make up the language of Geometry: a science which treats of space, by comparing portions of it with each other, for the purpose of pointing out their properties and mutual relations.

Analysis is a general term embracing that entire portion of mathematical science in which the quantities considered are represented by letters of the alphabet, and the operations to be performed on them are indicated by signs.

Algebra, which is a branch of Analysis, is also a species of universal arithmetic, in which letters and signs are employed to abridge and generalize all processes involving numbers. It is divided into two parts, corresponding to the science and art of Arithmetic:

1st. That which has for its object the investigation of the properties of numbers, embracing all the processes of reasoning, by which new properties are inferred from known ones; and,

2d. The solution of all problems or questions involving the determination of certain numbers which are unknown, from their connection with certain others which are known or given.

In arithmetic, all quantity is regarded as consisting of parts, which can be numbered exactly or approximatively, and in this respect, possesses all the properties of numbers. Propositions, therefore, concerning numbers, have this remarkable peculiarity, that they are propositions concerning all quantities whatever. Algebra extends the generalization still further. A number is a collection of things of the same kind, without reference to the nature of the thing, and is generally expressed by figures. Algebraic symbols may stand for *all numbers*, or for all quantities which numbers represent, or even for quantities which cannot be exactly expressed numerically.

In Geometry, each geometrical figure stands for a class; and when we have demonstrated a property of a figure, that property is considered proved for every figure of the class. In Algebra, all numbers, all lines, all surfaces, all solids, may be denoted by a single symbol, a or x. Hence, the conclusions deduced by means of those symbols are true of all things whatever, and not like those of number and Geometry, true only for particular classes of things. The symbols of Algebra, therefore, should not excite in our minds ideas of particular things. The written characters, a, b, c, d, x, y, z, serve as the representatives of things in general, whether abstract or concrete, whether known or unknown, whether finite or infinite.

In the various uses which we make of these symbols, and the processes of reasoning carried on by means of them, the mind insensibly comes to regard them as *things*, and not as mere signs; and we constantly predicate of them the properties of things in general, without pausing to inquire what kind of

thing is implied. All this we are at liberty to do, since the symbols being the representatives of quantity in general, there is no necessity of keeping the idea of *quantity* continually alive in the mind; and the processes of thought may, without danger, be allowed to rest on the symbols themselves, and therefore, become to that extent, merely mechanical. But when we look back and see on what the reasoning is based, and how the processes have been conducted, we shall find that every step was taken on the supposition that we were actually dealing with *things*, and not with *symbols;* and that without this understanding of the language, the whole system is without signification, and fails.*

The quantities which are the subjects of the algebraic analysis may be divided into two classes: those which are known or given, and those which are unknown or sought. The known are uniformly represented by the first letters of the alphabet, a, b, c, d, &c.; and the unknown by the final letters, x, y, z, v, &c.

Five operations, only, can be performed upon a quantity that will give results differing from the quantity itself: viz.

1st. To add a quantity to it;

2d. To subtract a quantity from it;

3d. To multiply it by a quantity;

4th. To divide it;

5th. To extract a root of it.

Five signs only, are employed to denote these operations. They are too well known to be repeated here. These, with

* Davies' Logic and Utility of Mathematics. § 278.

the signs of equality and inequality, together with the letters of the alphabet, are the elements of the algebraic language.

The interpretation of the language of Algebra is the first thing to which the attention of a pupil should be directed; and he should be drilled in the meaning and import of the symbols, until their significations and uses are as familiar as the sounds of the letters of the alphabet.

All the apprehensions, or elementary ideas, are conveyed to the mind by means of definitions and arbitrary signs; and every judgment is the result of a comparison of such impressions. Hence, the connection between the symbols and the ideas which they stand for, should be so close and intimate, that the one shall always suggest the other; and thus, the processes of Algebra become chains of thought, in which each link fulfils the double office of a distinct and connecting proposition.

ELEMENTS OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND PRELIMINARY REMARKS.

1. Quantity is anything which can be increased or diminished, and measured.

2. Mathematics is the science which treats of the measurement and relations of quantities.

3. Algebra is a branch of mathematics, in which the quantities considered are represented by letters, and the operations to be performed upon them are indicated by signs. The letters and signs are called *symbols*.

4. In algebra two kinds of quantities are considered:

1st. *Known quantities*, or those whose values are known or given. These are represented by the leading letters of the alphabet, as, a, b, c, &c.

2d. *Unknown quantities*, or those whose values are not given. They are denoted by the final letters of the alphabet, as, x, y, z, &c.

Letters employed to represent quantities are sometimes written with one or more dashes, as, a', b'', c''', x', y'', &c., and are read, a *prime*, b *second*, c *third*, x *prime*, y *second*, &c.

5. The sign $+$, is called *plus*, and when placed between two quantities, indicates that the one on the right is to be added to the one on the left. Thus, $a + b$ is read a plus b, and indicates

that the quantity represented by b is to be added to the quantity represented by a.

6. The sign $-$, is called *minus*, and when placed between two quantities, indicates that the one on the right is to be subtracted from the one on the left. Thus, $c - d$ is read c minus d, and indicates that the quantity represented by d is to be subtracted from the quantity represented by c.

The sign $+$, is sometimes called the *positive* sign, and the quantity before which it is placed is said to be *positive*.

The sign $-$, is called the *negative* sign, and quantities affected by it are said to be *negative*.

7. The sign $\times$, is called the sign of multiplication, and when placed between two quantities, indicates that the one on the left is to be multiplied by the one on the right. Thus, $a \times b$, indicates that a is to be multiplied by b. The multiplication of quantities may also be indicated by placing a simple point between them, as $a.b$, which is read a multiplied by b.

The multiplication of quantities, which are represented by letters, is generally indicated by simply writing the letters one after another, without interposing any sign. Thus,

ab is the same as $a \times b$, or $a.b$;

and abc, the same as $a \times b \times c$, or $a.b.c$.

It is plain that the notation last explained cannot be employed when the quantities are represented by figures. For, if it were required to indicate that 5 was to be multiplied by 6, we could not write 5 6, without confounding the product with the number 56.

The result of a multiplication is called the *product*, and each of the quantities employed, is called a *factor*. In the product of several letters, each single letter is called a *literal factor*. Thus, in the product ab there are two literal factors a and b; in the product bcd there are three, b, c and d.

8. The sign $\div$, is called the sign of division, and when placed between two quantities, indicates that the one on the left is to be divided by the one on the right. Thus, $a \div b$ indicates that a is to

be divided by b. The same operation may be indicated by writing b under a, and drawing a line between them, as $\frac{a}{b}$; or by writing b on the right of a, and drawing a line between them, as $a|\underline{b}$.

9. The sign $=$, is called the sign of *equality*, and indicates that the two quantities between which it is placed are equal to each other. Thus, $a - b = c + d$, indicates that a diminished by b is equal to c increased by d.

10. The sign $>$, is called the sign of *inequality*, and is used to indicate that one quantity is greater or less than another.

Thus, $a > b$ is read, *a greater* than b; and $a < b$ is read, *a less* than b; that is, the opening of the sign is turned toward the greater quantity.

11. The sign $\sim$ is sometimes employed to indicate the difference of two quantities when it is not known which is the greater.

Thus, $a \sim b$, indicates the *difference* between a and b, without showing which is to be subtracted from the other.

12. The sign $\propto$, is used to indicate that one quantity varies as to another. Thus $a \propto \frac{1}{b}$, indicates that a varies as $\frac{1}{b}$.

13. The signs : and : :, are called the signs of proportion; the first is read, *is to*, and the second is read, *as*. Thus,

$$a : b :: c : d,$$

is read, *a is to b, as c is to d.*

The sign $\therefore$, is read *hence*, or *consequently*.

14. If a quantity is taken several times, as

$$a + a + a + a + a,$$

it is generally written but once, and a number is then placed before it, to show how many times it is taken. Thus,

$$a + a + a + a + a \text{ may be written } 5a.$$

The number 5 is called the *co-efficient* of a, and denotes that a is taken 5 times.

Hence, a *co-efficient* is a number prefixed to a quantity denoting the number of times which the quantity is taken.

When no co-efficient is written, the co-efficient 1 is always understood; thus, a is the same as $1a$.

15. If a quantity is taken several times as a factor, the product may be expressed by writing the quantity once, and placing a number to the right and above it, to show how many times it is taken as a factor.

Thus, $a \times a \times a \times a \times a$ may be written a^5.

The number 5 is called an *exponent*, and indicates that a is taken 5 times as a factor.

Hence, an *exponent* is a number written to the right and above a quantity, to show how many times it is taken as a factor. If no exponent is written, the exponent 1 is understood. Thus, a is the same as a^1.

16. If a quantity be taken any number of times as a factor, the resulting product is called a *power* of that quantity: the exponent denotes the *degree* of the power. For example,

$a^1 = a$ is the first power of a,

$a^2 = a \times a$ is the second power, or square of a,

$a^3 = a \times a \times a$ is the third power, or cube of a,

$a^4 = a \times a \times a \times a$ is the fourth power of a,

$a^5 = a \times a \times a \times a \times a$ is the fifth power of a,

in which the exponents of the powers are, 1, 2, 3, 4 and 5; and the powers themselves, are the results of the multiplications. It should be observed that the *exponent of a power* is always greater by *one* than the number of multiplications. The exponent of a *power* of a quantity is sometimes, for the sake of brevity, called *the exponent of the quantity.*

17. As an example of the use of the exponent in algebra, let it be required to express that a number a is to be multiplied three times by itself; that this product is then to be multiplied three times by b, and this new product twice by c; we should write

$$a \times a \times a \times a \times b \times b \times b \times c \times c = a^4b^3c^2.$$

If it were further required to take this result a certain number of times, say seven, we should simply write $7a^4b^3c^2$

18. A *root* of a quantity, is a quantity which being taken a certain number of times, as a factor, will produce the given quantity.

The sign $\sqrt{\ }$, is called the *radical sign*, and when placed over a quantity, indicates that its root is to be extracted. Thus,

$\sqrt[2]{a}$ or simply $\sqrt{a}$ denotes the square root of a.

$\sqrt[3]{a}$ denotes the cube root of a.

$\sqrt[4]{a}$ denotes the fourth root of a.

The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, &c.

19. The *reciprocal* of a quantity, is 1 divided by that quantity. Thus,

$\dfrac{1}{a}$ is the reciprocal of a;

and $\dfrac{1}{a+b}$ is the reciprocal of $a+b$.

20. Every quantity written in algebraic language, that is, by the aid of letters and signs, is called an *algebraic quantity*, or the *algebraic expression* of a quantity. Thus,

$3a$ is the algebraic expression of three times the quantity denoted by a;

$5a^2$ is the algebraic expression of five times the square of a;

$7a^3b^2$ is the algebraic expression of seven times the product of the cube of a and the square of b;

$3a-5b$ is the algebraic expression of the difference between three times a and five times b;

$2a^2-3ab+4b^2$ is the algebraic expression of twice the square of a, diminished by three times the product of a and b, augmented by four times the square of b.

21. A single algebraic expression, not connected with any other by the sign of addition or subtraction, is called a *monomial*, or simply, a *term*.

Thus, $3a$, $5a^2$, $7a^3b^2$, are monomials, or single terms.

An algebraic expression composed of two or more terms connected by the sign $+$ or $-$, is called a *polynomial.*

For example, $3a - 5b$ and $2a^2 - 3cb + 4b^2$, are polynomials.

A polynomial of two terms, is called a *binomial;* and one of three terms, a *trinomial.*

22. The *numerical value* of an algebraic expression, is the number obtained by giving a particular value to each letter which enters it, and performing the operations indicated. This numerical value will depend on the particular values attributed to the letters, and will *generally* vary with them.

For example, the numerical value of $2a^3$, will be 54 if we make $a = 3$; for, $3^3 = 3 \times 3 \times 3 = 27$, and $2 \times 27 = 54$.

The numerical value of the same expression is 250 when we make $a = 5$; for, $5^3 = 5 \times 5 \times 5 = 125$, and $2 \times 125 = 250$.

We say that the numerical value of an algebraic expression *generally* varies with the values of the letters which enter it; it does not, however, always do so. Thus, in the expression $a - b$, so long as a and b are increased or diminished by the same number, the value of the expression will not be changed.

For example, make $a = 7$ and $b = 4$: there results $a - b = 3$.

Now, make $a = 7 + 5 = 12$, and $b = 4 + 5 = 9$, and there results, as before, $a - b = 12 - 9 = 3$.

23. Of the different terms which compose a polynomial, some are preceded by the sign $+$, and others by the sign $-$. The former are called *additive terms*, the latter, *subtractive terms.*

When the first term of a polynomial is plus, the sign is generally omitted; and when no sign is written before a term, it is always understood to have the sign $+$.

24. The *numerical value* of a polynomial is not affected by changing the *order* of its terms, provided the signs of all the terms remain unchanged. For example, the polynomial

$$4a^3 - 3a^2b + 5ac^2 = 5ac^2 - 3a^2b + 4a^3 = -3a^2b + 5ac^2 + 4a^3.$$

25. Each literal factor which enters a term, is called a *dimension* of the term; and the *degree* of a term is indicated by the number of these factors or dimensions. Thus

$3a$ is a term of one dimension, or of the first degree.

$5ab$ is a term of two dimensions, or of the second degree.

$7a^3bc^2 = 7aaabcc$ is of six dimensions, or of the sixth degree.

In general, *the degree of a term is determined by taking the sum of the exponents of the letters which enter it.* For example, the term $8a^2bcd^3$ is of the seventh degree, since the sum of the exponents,

$$2 + 1 + 1 + 3, \text{ is equal to } 7.$$

26. A polynomial is said to be *homogeneous*, when all of its terms are of the same degree. The polynomial

$3a - 2b + c$ is homogeneous and of the first degree.

$-4ab + b^2$ is homogeneous and of the second degree.

$5a^2c - 4c^3 + 2c^2d$ is homogeneous and of the third degree.

$8a^3 - 4ab + c$ is not homogeneous.

27. A vinculum ——, parenthesis ( ), brackets [ ], { }, or bar |, may be used to indicate that all the quantities which they connect are to be considered together. Thus,

$\overline{a + b + c} \times x$, $(a + b + c) \times x$, $[a + b + c] \times x$, or $\{a + b + c\}x$,

indicate that the trinomial $a + b + c$ is to be multiplied by x.

When the parenthesis or brackets are used, the sign of multiplication may be omitted: as, $(a + b + c)x$. The bar is used in some cases, and differs from the vinculum in being placed vertically, as

$$\left.\begin{array}{r} +a \\ +b \\ +c \end{array}\right| x.$$

28. Terms which contain the same letters affected with equal exponents are said to be *similar*. Thus, in the polynomial,

$$7ab + 3ab - 4a^3b^2 + 5a^3b^2,$$

the terms $7ab$ and $3ab$, are similar, and so also are the terms $-4a^3b^2$ and $5a^3b^2$, the letters in each being the same, and the same letters being affected with equal exponents. But in the binomial

$$8a^2b + 7ab^2,$$

the terms are not similar; for, although they contain the same letters, yet the same letters are not affected with equal exponents.

29. When a polynomial contains similar terms, it may be reduced to a simpler form by forming a single term from each set of similar terms. It is said to be in its *simplest form*, when it contains the fewest terms to which it can be reduced.

If we take the polynomial

$$2a^3bc^2 - 4a^3bc^2 + 6a^3bc^2 - 8a^3bc^2 + 11a^3bc^2,$$

we know, from the definition of a co-efficient, that the literal part a^3bc^2 is to be taken additively, $2 + 6 + 11$, or 19 times; and subtractively, $4 + 8$, or 12 times.

Hence, the given polynomial reduces to

$$19a^3bc^2 - 12a^3bc^2 = 7a^3bc^2.$$

It may happen that the co-efficient of the subtractive term, obtained as above, will exceed that of the additive term. In that case, *subtract the positive co-efficient from the negative, prefix the minus sign to the remainder, and then annex the literal part.*

In the polynomial

$$3a^2b + 2a^2b - 5a^2b - 3a^2b,$$

we have,

$$\begin{array}{rcr} +3a^2b & \qquad & -5a^2b \\ +2a^2b & & -3a^2b \\ \hline +5a^2b & & -8a^2b \end{array}$$

But, $\quad -8a^2b = -5a^2b - 3a^2b$: hence

$$5a^2b - 8a^2b = 5a^2b - 5a^2b - 3a^2b = -3a^2b.$$

In like manner we may reduce the similar terms of any polynomial. Hence, for the reduction of a polynomial containing sets of similar terms, to its simplest form, we have the following

RULE.

I. *Add together the co-efficients of all the additive terms of each set, and annex to their sum the literal part: form a single subtractive term in the same manner.*

II. *Then, subtract the less co-efficient from the greater, and to the remainder prefix the sign of the greater co-efficient, and annex the literal part.*

EXAMPLES.

1. Reduce the polynomial $4a^2b - 8a^2b - 9a^2b + 11a^2b$ to its simplest form. *Ans.* $-2a^2b$.

2. Reduce the polynomial $7abc^2 - abc^2 - 7abc^2 - 8abc^2 + 6abc^2$ to its simplest form. *Ans.* $-3abc^2$.

3. Reduce the polynomial $9cb^3 - 8ac^2 + 15cb^3 + 8ca + 9ac^2 - 24cb^3$ to its simplest form. *Ans.* $ac^2 + 8ca$.

4. Reduce the polynomial $6ac^2 - 5ab^3 + 7ac^2 - 3ab^3 - 13ac^2 + 18ab^3$ to its simplest form. *Ans.* $10ab^3$.

5. Reduce the polynomial $abc^2 - abc + 5ac^2 - 9abc^2 + 6abc - 8ac^2$ to its simplest form. *Ans.* $-8abc^2 + 5abc - 3ac^2$.

6. Reduce the polynomial $3a^2b^2 - 7a^3b + 5ab - 9a^2b^2 + 9a^3b + 3ab$ to its simplest form. *Ans.* $-6a^2b^2 + 2a^3b + 8ab$.

7. Reduce the polynomial $3acb^4 - 7a^3c^2b^3 - 6a^4b^5 - 3acb^4 + 6a^3c^2b^3 - 6acb^4 + 4a^4b^5 + 2a^4b^5$ to its simplest form.

Ans. $-a^3c^2b^3 - 6acb^4$.

8. Reduce the polynomial $-7a^2b^2c^2 + 9a^5bc^2 + 6a^2b^2c^2 + a^2b^2c^2 - 5a^5bc^2 - b^5c^5$ to its simplest form. *Ans.* $4a^5bc^2 - b^5c^5$.

9. Reduce the polynomial $-10a^3b + 6a^2b^2 + 7a^3b - 5a^2b^2 - 5a^3b + 3a^2b^2$ to its simplest form. *Ans.* $-8a^3b + 4a^2b^2$.

REMARK.—It should be observed that the reduction affects only the co-efficients, and not the exponents.

30. A THEOREM is a general truth, which is made evident by a course of reasoning called a demonstration.

A PROBLEM is a question proposed which requires a solution.

31. We shall now illustrate the utility and brevity of algebraic language by solving the following

PROBLEM.

The sum of two numbers is 67, *and their difference is* 19; *what are the numbers?*

Let us first indicate, by the aid of algebraic symbols, the relation which exists between the given and unknown numbers of the problem.

If the less of the two numbers were *known*, the greater could be found by adding to it the difference 19; or in other words, the less number, plus 19, is equal to the greater.

If, then, we denote the less number by x,

$x + 19$ will denote the greater,

and $2x + 19$ will denote the sum.

But from the enunciation, this sum is to be equal to 67. Therefore,

$$2x + 19 = 67.$$

Now, if $2x$ augmented by 19, is equal to 67, $2x$ alone is equal to 67 minus 19, or

$$2x = 67 - 19,$$

or performing the subtraction,

$$2x = 48.$$

Hence, x is equal to half of 48, that is

$$x = \frac{48}{2} = 24.$$

The less number being 24, the greater is

$$x + 19 = 24 + 19 = 43.$$

And, indeed, we have

$$43 + 24 = 67, \text{ and } 43 - 24 = 19.$$

GENERAL SOLUTION.

The sum of two numbers is a, and their difference is b. What are the two numbers?

Let x denote the less number;

Then will $x + b$ denote the greater number.

Now, from the conditions of the problem,

$$x + x + b, \text{ or } 2x + b$$

will be equal to the sum of the two numbers: hence,

$$2x + b = a.$$

Now, if $2x + b$ is equal to a, $2x$ alone must be equal to $a - b$ and

$$x = \frac{a - b}{2} = \frac{a}{2} - \frac{b}{2}.$$

If the value of x be increased by b, we shall have the greater number: that is,

$$x + b = \frac{a}{2} - \frac{b}{2} + b = \frac{a}{2} + \frac{b}{2};$$

hence, $\quad x + b = \dfrac{a}{2} + \dfrac{b}{2} =$ the greater number, and

$$x = \frac{a}{2} - \frac{b}{2} = \text{ the less number.}$$

That is, *the greater of two numbers is equal to half their sum increased by half their difference; and the less is equal to half their sum diminished by half their difference.*

As the *form* of these results is independent of any particular values attributed to the letters a and b, the expressions are called *formulas*, and may be regarded as comprehending the solution of all problems of the same kind, differing only in the numerical values of the given quantities. Hence,

A *formula* is the algebraic expression of a general rule, or principle.

To apply these formulas to the case in which the sum is 237 and difference 99, we have

the greater number $= \dfrac{237}{2} + \dfrac{99}{2} = \dfrac{237 + 99}{2} = \dfrac{336}{2} = 168;$

and the less $= \dfrac{237}{2} - \dfrac{99}{2} = \dfrac{237 - 99}{2} = \dfrac{138}{2} = 69;$

and these are the true numbers; for,

$168 + 69 = 237$ which is the given sum,

and $\quad 168 - 69 = 99$ which is the given difference.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

ADDITION.

31. ADDITION, in algebra, is the operation of finding the simplest equivalent expression for the aggregate of two or more algebraic quantities. Such equivalent expression is called their *sum.*

32. If the quantities to be added are dissimilar, no reductions can be made among the terms. We then write them one after the other, each with its proper sign, and the resulting polynomial will be the simplest expression for the sum.

For example, let it be required to add together the monomials

$$3a, \quad 5b \text{ and } 2c;$$

we connect them by the sign of addition,

$$3a + 5b + 2c,$$

a result which cannot be reduced to a simpler form.

33. If some of the quantities to be added have similar terms, we connect the quantities by the sign of addition as before, and then reduce the resulting polynomial to its simplest form, by the rule already given. This reduction will, in general, be more readily accomplished if we write down the quantities to be added, so that similar terms shall fall in the same column. Thus;

Let it be required to find the sum of the quantities,
$$\left\{\begin{array}{rrr} 3a^2 & - 4ab & \\ 2a^2 & - 3ab & + b^2 \\ & 2ab & - 5b^2 \end{array}\right.$$

Their sum, after reducing (Art. 29), is
$$\underline{5a^2 - 5ab - 4b^2}$$

34. As operations similar to the above apply to all algebraic expressions, we deduce, for the addition of algebraic quantities, the following general

RULE.

I. *Write down the quantities to be added, with their respective signs, so that the similar terms shall fall in the same column.*

II. *Reduce the similar terms, and annex to the results those terms which cannot be reduced, giving to each term its respective sign.*

EXAMPLES.

1. Add together the polynomials,

$$3a^2 - 2b^2 - 4ab,\quad 5a^2 - b^2 + 2ab \text{ and } 3ab - 3c^2 - 2b^2.$$

The term $3a^2$ being similar to $5a^2$ we write $8a^2$ for the result of the reduction of these two terms, at the same time slightly crossing them as in the terms of the example.

$$\begin{array}{rrrr} \cancel{3a^2} & -\cancel{4ab} & -\cancel{2b^2} & \\ \cancel{5a^2} & +\cancel{2ab} & -\cancel{b^2} & \\ & +\cancel{3ab} & -\cancel{2b^2} & \\ \hline 8a^2 & +ab & -5b^2 & -3c^2 \end{array}$$

Passing then to the term $-4ab$, which is similar to the two terms $+2ab$ and $+3ab$, the three reduce to $+ab$, which is placed after $8a^2$, and the terms crossed like the first term. Passing then to the terms involving b^2, we find their sum to be $-5b^2$, after which we write $-3c^2$.

The marks are drawn across the terms, that none of them may be overlooked and omitted.

(2).

$$\begin{array}{lrrr} & 7x & +3ab & +2c \\ & -3x & -3ab & -5c \\ & 5x & -9ab & -9c \\ \hline \text{Sum} \ldots & 9x & -9ab & -12c \end{array}$$

(3).

$$\begin{array}{rrr} 16a^2b^2 & +bc & -2abc \\ -4a^2b^2 & -9bc & +6abc \\ -9a^2b^2 & +bc & +abc \\ \hline 3a^2b^2 & -7bc & +5abc \end{array}$$

(4).

$$\begin{array}{lrrrr} & a & +ab & -cd & +f \\ & 3a & +5ab & -6cd & -f \\ & -5a & -6ab & +6cd & -7f \\ & -a & +ab & +cd & +4f \\ \hline \text{Sum} & -2a & +ab & +0 & -3f \end{array}$$

(5).

$$\begin{array}{rrrr} 6ab & +cd & +d & \\ 3ab & +5cd & -y & \\ -4ab & +6cd & +x & \\ -5ab & -12cd & +y & \\ \hline 0 & 0 & +x & +d \end{array}$$

6. Add together $3a + b$, $3a + 3b$, $-9a - 7b$, $6a + 9b$ and $8a + 3b + 8c$. *Ans.* $11a + 9b + 8c$.

7. Add together $3ax + 3ac + f$, $-9ax + 7a + d$, $+6ax + 3ac + 3f$, $8ax + 13ac + 9f$ and $-14f + 3ax$.

Ans. $11ax + 19ac - f + 7a + d$.

8. Add together the polynomials, $3a^2c + 5ab$, $7a^2c - 3ab + 3ac$ $5a^2c - 6ab + 9ac$ and $-8a^2c + ab - 12ac$. *Ans.* $7a^2c - 3ab$.

9. Add the polynomials, $19a^2x^3b - 12a^3cb$, $5a^2x^3b + 15a^3cb - 10ax$, $-2a^2x^3b - 13a^3cb$ and $-18a^2x^3b - 12a^3cb + 9ax$.

Ans. $4a^2x^3b - 22a^3cb - ax$.

10. Add together $3a + b + c$, $5a + 2b + 3ac$, $a + c + ac$ and $-3a - 9ac - 8b$. *Ans.* $6a - 5b + 2c - 5ac$.

11. Add together $5a^2b + 6cx + 9bc^2$, $7cx - 8a^2b$ and $-15cx - 9bc^2 + 2a^2b$. *Ans.* $-a^2b - 2cx$.

12. Add together $8ax + 5ab + 3a^2b^2c^2$, $-18ax + 6a^2 + 10ab$ and $10ax - 15ab - 6a^2b^2c^2$. *Ans.* $-3a^2b^2c^2 + 6a^2$.

13. What is the sum of $41a^3b^2c - 27abc - 14a^2y$ and $10a^3b^2c + 9abc$? *Ans.* $51a^3b^2c - 18abc - 14a^2y$.

14. What is the sum of $18abc - 9ab + 6c^2 - 3c + 9ax$ and $9abc + 3c - 9ax$? *Ans.* $27abc - 9ab + 6c^2$.

15. What is the sum of $8abc + b^3a - 2cx - 6xy$ and $7cx - xy - 13b^3a$? *Ans.* $8abc - 12b^3a + 5cx - 7xy$.

16. What is the sum of $9a^2c - 14aby + 15a^2b^2$ and $-a^2c - 8a^2b^2$? *Ans.* $8a^2c - 14aby + 7a^2b^2$.

17. What is the sum of $17a^5b^2 + 9a^3b - 3a^2$, $-14a^5b^2 + 7a^3 - 9a^3$, $-15a^3b + 7a^5b^2 - a^3$ and $14a^3b - 19a^3b$?

Ans. ————————.

18. What is the sum of $3ax^2 - 9ax^3 - 17axy$, $+9ax^2 + 18ax^3 + 34axy$ and $7a^5b + 3ax^3 - 7ax^2 + 4bcx$? *Ans.* ————————.

19. Add together $3a^2 + 5a^2b^2c^2 - 9a^3x$, $7a^2 - 8a^2b^2c^2 - 10a^3x$ and $10ab + 16a^2b^2c^2 + 19a^3x$. *Ans.* $10a^2 + 13a^2b^2c^2 + 10ab$.

20. Add together $7a^2b - 3abc - 8b^2c - 9c^3 + cd^2$, $8abc - 5a^2b + 3c^3 - 4b^2c + cd^2$, and $4a^2b - 8c^3 + 9b^2c - 3d^3$.

Ans. $6a^2b + 5abc - 3b^2c - 14c^3 + 2cd^2 - 3d^3$.

21. Add together $-18a^3b+2ab^4+6a^2b^2$, $-8ab^4+7a^3b-5a^2b^2$ and $-5a^3b+6ab^4+11a^2b^2$. *Ans.* $-16a^3b+12a^2b^2$.

22. What is the sum of $3a^3b^2c-16a^4x-9ax^3d$, $+6a^3b^2c-6ax^3d+7a^4x$ and $+16ax^3d-a^4x-8a^3b^2c$?

Ans. $a^3b^2c+ax^3d$.

23. What is the sum of the following terms: viz., $8a^5-10a^4b-16a^3b^2+4a^2b^3-12a^4b+15a^3b^2+24a^2b^3-6ab^4-16a^3b^2+20a^2b^3+32ab^4-8b^5$?

Ans. $8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5$.

SUBTRACTION.

35. SUBTRACTION, in algebra, is the operation for finding the simplest expression for the difference between two algebraic quantities. This difference is called the *remainder*.

36. Let it be required to subtract $4b$ from $5a$. Here, as the quantities are not similar, their difference can only be indicated, and we write

$$5a-4b.$$

Again, let it be required to subtract $4a^3b$ from $7a^3b$. These terms being similar, one of them may be taken from the other and their true difference is expressed by

$$7a^3b-4a^3b=3a^3b.$$

37. Generally, if from one polynomial we wish to subtract another, the operation may be indicated by enclosing the second in a parenthesis, prefixing the minus sign, and then writing it after the first. To deduce a rule for performing the operation thus indicated, let us represent the sum of all the terms in the first polynomial by a. Let c represent the sum of all the additive terms in the other polynomial, and $-d$ the sum of the subtractive terms; then this polynomial will be represented by $c-d$. The operation may then be indicated thus,

$$a-(c-d);$$

where it is required to subtract from a the difference between c and d.

If, now, we diminish the quantity a by the quantity c, the result $a - c$ will be too small by the quantity d, since c should have been diminished by d before taking it from a. Hence, to obtain the true remainder, we must increase the first result by d, which gives the expression

$$a - c + d,$$

and this is the true remainder.

By comparing this remainder with the given polynomials, we see that we have changed the signs of all the terms of the quantity to be subtracted, and added the result to the other quantity. To facilitate the operation, similar quantities are written in the same column.

Hence, for the subtraction of algebraic quantities, we have the following

RULE.

I. *Write the quantity to be subtracted under that from which it is to be taken, placing the similar terms, if there are any, in the same column.*

II. *Change the signs of all the terms of the quantity to be subtracted, or conceive them to be changed, and then add the result to the other quantity.*

EXAMPLES.

(1).

From -	$6ac - 5ab + c^2$
Take -	$3ac + 3ab - 7c$
Remainder	$3ac - 8ab + c^2 + 7c.$

The same with the signs of the lower line changed.

(1).

$$\begin{array}{l} 6ac - 5ab + c^2 \\ -3ac - 3ab + 7c \\ \hline 3ac - 8ab + c^2 + 7c. \end{array}$$

(2).

From -	$16a^2 - 5bc + 7ac$
Take -	$14a^2 + 5bc + 8ac$
Remainder	$2a^2 - 10bc - ac$

(3).

$$\begin{array}{l} 19abc - 16ax - 5axy \\ 17abc + 7ax - 15axy \\ \hline 2abc - 23ax + 10axy \end{array}$$

(4).

From -	$5a^3 - 4a^2b + 3b^2c$
Take -	$-2a^3 + 3a^2b - 8b^2c$
Remainder	$7a^3 - 7a^2b + 11b^2c$

(5).

$$\begin{array}{l} 4ab - cd + 3a^2 \\ 5ab - 4cd + 3a^2 + 5b^2 \\ \hline -ab + 3cd + 0 - 5b^2. \end{array}$$

6. From $3a^2x - 13abc + 7a^2$, take $9a^2x - 13abc$.
Ans. $-6a^2x + 7a^2$.

7. From $51a^3b^2c - 18abc - 14a^2y$, take $41a^3b^2c - 27abc - 14a^2y$.
Ans. $10a^3b^2c + 9abc$.

8. From $27abc - 9ab + 6c^2$, take $9abc + 3c - 9ax$.
Ans. $18abc - 9ab + 6c^2 - 3c + 9ax$.

9. From $8abc - 12b^3a + 5cx - 7xy$, take $7cx - xy - 13b^3a$.
Ans. $8abc + b^3a - 2cx - 6xy$.

10. From $8a^2c - 14aby + 7a^2b^2$, take $9a^2c - 14aby + 15a^2b^2$.
Ans. $-a^2c - 8a^2b^2$.

11. From $9a^6x^2 - 13 + 20ab^3x - 4b^6cx^2$, take $3b^6cx^2 + 9a^6x^2 - 6 + 3ab^3x$.
Ans. $17ab^3x - 7b^6cx^2 - 7$.

12. From $5a^4 - 7a^3b^2 - 3c^5d^2 + 7d$, take $3a^4 - 3a^2 - 7c^5d^2 - 15a^3b^2$.
Ans. $2a^4 + 8a^3b^2 + 4c^5d^2 + 7d + 3a^2$.

13. From $51a^2b^2 - 48a^3b + 10a^4$, take $10a^4 - 8a^3b - 6a^2b^2$.
Ans. $57a^2b^2 - 40a^3b$.

14. From $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5$, take $64x^2y^3 + 48xy^4 - 40y^5$.
Ans. $20xy^4 - 39x^2y^3 + 21x^3y^2$.

15. From $53x^3y^2 - 15x^2y^3 - 18x^4y - 56x^5$, take $-15x^2y^3 + 18x^3y^2 + 24x^4y$.
Ans. $35x^3y^2 - 42x^4y - 56x^5$.

38. From what has preceded, we see that polynomials may be subjected to certain transformations.

For example - - - - $6a^2 - 3ab + 2b^2 - 2bc$,
may be written - - - - $6a^2 - (3ab - 2b^2 + 2bc)$.
In like manner - - - - $7a^3 - 8a^2b - 4b^2c + 6b^3$,
may be written - - - - $7a^3 - (8a^2b + 4b^2c - 6b^3)$;
or, again, - - - - - - $7a^3 - 8a^2b - (4b^2c - 6b^3)$.
Also, - - - - - - - - $8a^2 - 6a^2b^2 + 5a^2b^3$,
becomes - - - - - - - $8a^2 - (6a^2b^2 - 5a^2b^3)$.
Also, - - - - - - - - $9a^2c^3 - 8a^4 + b^2 - c$.
may be written - - - - $9a^2c^3 - (8a^4 - b^2 + c)$;
or, it may be written - - $9a^2c^3 + b^2 - (8a^4 + c)$.

These transformations consist in separating a polynomial into two parts, and then connecting the parts by the minus sign.

It will be observed that the sign of each term is changed when the term is placed within the parenthesis. Hence, if we have one or more terms included within a parenthesis having the minus sign before it, *the signs of all the terms must be changed* when the parenthesis is omitted.

Thus, $4a - (6ab - 3c - 2b)$,
is equal to $4a - 6ab + 3c + 2b$.
Also, $6ab - (-4ac + 3d - 4ab)$,
is equal to $6ab + 4ac - 3d + 4ab$.

39. REMARK.—From what has been shown in addition and subtraction, we deduce the following principles.

1st. In Algebra, the words *add* and *sum* do not always, as in arithmetic, convey the idea of augmentation. For, if to a we add $-b$, the sum is expressed by $a - b$, and this is, properly speaking, the arithmetical difference between the number of units expressed by a, and the number of units expressed by b. Consequently, this result is actually less than a.

To distinguish this sum from an arithmetical sum, it is called the *algebraic sum.*

Thus, the polynomial, $2a^3 - 3a^2b + 3b^2c$,

is an algebraic sum, so long as it is considered as the result of the union of the monomials

$$2a^3, \quad -3a^2b, \quad +3b^2c,$$

with their respective signs; but, in its *proper acceptation*, it is the arithmetical difference between the sum of the units contained in the additive terms, and the units contained in the subtractive term.

It follows from this, that an algebraic sum may, in the numerical applications, be reduced to a *negative* expression.

2d. The words *subtraction* and *difference*, do not always convey the idea of diminution. For, the difference between $+a$ and $-b$ being

$$a - (-b) = a + b,$$

is numerically greater than a. This result is an *algebraic difference.*

40. It frequently occurs in Algebra, that the *algebraic* sign $+$ or $-$, which is written, is not the *true* sign of the term before which it is placed. Thus, if it were required to subtract $-b$ from a, we should write

$$a - (-b) = a + b.$$

Here the *true* sign of the second term of the binomial is plus, although its *algebraic* sign is $-$. This minus sign, operating upon the sign of b, which is also negative, produces a plus sign for b in the result. The sign which results, after combining the algebraic sign with the sign of the quantity, *is called the essential sign of the term*, and is often different from the algebraic sign.

MULTIPLICATION.

41. Multiplication, in Algebra, is the operation of finding the product of two algebraic quantities. The quantity to be multiplied is called the *multiplicand;* the quantity by which it is multiplied is called the *multiplier;* and both are called *factors.*

42. Let us first consider the case in which both factors are monomials.

Let it be required to multiply $7a^3b^2$ by $4a^2b$; the operation may be indicated thus,

$$7a^3b^2 \times 4a^2b,$$

or by resolving both multiplicand and multiplier into their simple factors,

$$7aaabb \times 4aab.$$

Now, it has been shown in arithmetic, that the value of a product is not changed by changing the order of its factors; hence, we may write the product as follows:

$$7 \times 4aaaaabbb, \text{ which is equivalent to } 28a^5b^3.$$

Comparing this result with the given factors, we see that the co-efficient in the product is equal to the product of the co-efficients of the multiplicand and multiplier; and that the exponent of each letter is equal to the sum of the exponents of that letter in both multiplicand and multiplier.

And since the same course of reasoning may be applied to any two monomials, we have, for the multiplication of monomials, the following

RULE.

I. *Multiply the co-efficients together for a new co-efficient.*

II. *Write after this co-efficient all the letters which enter into the multiplicand and multiplier, giving to each an exponent equal to the sum of its exponents in both factors.*

EXAMPLES.

(1) - - $8a^2bc^2 \times 7abd^2 = 56a^3b^2c^2d^2$.

(2) - - $21a^3b^2dc \times 8abc^3 = 168a^4b^3c^4d$.

	(3)	(4)	(5)	(6)
Multiply - -	$3a^2b$	$12a^2x$	$6xyz$	a^2xy
by - -	$2ba^2$	$12x^2y$	ay^2z	$2xy^2$
	$6a^4b^2$	$144a^2x^3y$	$6axy^3z^2$	$2a^2x^2y^3$.

7. Multiply $8a^5b^2c$ by $7a^8b^5cd$. *Ans.* $56a^{13}b^7c^2d$.

8. Multiply $5abd^3$ by $12cd^5$. *Ans.* $60abcd^8$.

9. Multiply $7a^4bd^2c^3$ by $abdc$. *Ans.* $7a^5b^2d^3c^4$.

43. We will now proceed to the multiplication of polynomials. In order to explain the most general case, we will suppose the multiplicand and multiplier each to contain additive and subtractive terms.

Let a represent the sum of all the additive terms of the multiplicand, and $-b$ the sum of the subtractive terms; c the sum of the additive terms of the multiplier, and $-d$ the sum of the subtractive terms. The multiplicand will then be represented by $a - b$ and the multiplier, by $c - d$.

We will now show how the multiplication expressed by $(a - b) \times (c - d)$ can be effected.

$$\begin{array}{l} a - b \\ \underline{c - d} \\ ac - bc \\ \underline{\quad\;\; - ad + bd} \\ \underline{ac - bc - ad + bd.} \end{array}$$

The required product is equal to $a - b$ taken as many times as there are units in $c - d$. Let us first multiply by c; that is, take $a - b$ as many times as there are units in c. We begin by writing ac, which is too great by b taken

c times; for it is only the *difference* between a and b, that is first to be multiplied by c. Hence, $ac - bc$ is the product of $a - b$ by c.

But the true product is $a - b$ taken $c - d$ times: hence, the last product is too great by $a - b$ taken d times; that is, by $ad - bd$, which must, therefore, be subtracted. Subtracting this from the first product (Art. 37), we have

$$(a - b) \times (c - d) = ac - bc - ad + bd:$$

If we suppose a and c each equal to 0, the product will reduce to $+ bd$.

44. By considering the product of $a - b$ by $c - d$, we may deduce the following rule for signs, in multiplication.

When two terms of the multiplicand and multiplier are affected with the same sign, their product will be affected with the sign $+$, and when they are affected with contrary signs, their product will be affected with the sign $-$.

We say, in algebraic language, that $+$ multiplied by $+$ or $-$ multiplied by $-$, gives $+$; $-$ multiplied by $+$, or $+$ multiplied by $-$, gives $-$. But since mere signs cannot be multiplied together, this last enunciation does not, in itself, express a distinct idea, and should only be considered as an abbreviation of the preceding.

This is not the only case in which algebraists, for the sake of brevity, employ expressions in a technical sense in order to secure the advantage of fixing the rules in the memory.

45. We have, then, for the multiplication of polynomials, the following

RULE.

Multiply all the terms of the multiplicand by each term of the multiplier in succession, affecting the product of any two terms with the sign plus, when their signs are alike, and with the sign minus, when their signs are unlike. Then reduce the polynomial result to its simplest form.

EXAMPLES.

1. Multiply - - - - - - - - - $3a^2 + 4ab + b^2$

by - - - - - - - - - $2a + 5b$

$$\begin{array}{lr}
& 6a^3 + 8a^2b + 2ab^2 \\
& + 15a^2b + 20ab^2 + 5b^3 \\
\text{Product} & \overline{6a^3 + 23a^2b + 22ab^2 + 5b^3}
\end{array}$$

(2).

$$\begin{array}{l}
x^2 + y^2 \\
x - y \\
\hline
x^3 + xy^2 \\
\quad - x^2y - y^3 \\
\hline
x^3 + xy^2 - x^2y - y^3
\end{array}$$

(3).

$$\begin{array}{l}
x^5 + xy^6 + 7ax \\
ax + 5ax \\
\hline
ax^6 + ax^2y^6 + 7a^2x^2 \\
\quad + 5ax^6 + 5ax^2y^6 + 35a^2x^2 \\
\hline
6ax^6 + 6ax^2y^6 + 42a^2x^2.
\end{array}$$

4. Multiply $x^2 + 2ax + a^2$ by $x + a$.
Ans. $x^3 + 3ax^2 + 3a^2x + a^3$.

5. Multiply $x^2 + y^2$ by $x + y$.
Ans. $x^3 + xy^2 + x^2y + y^3$.

6. Multiply $3ab^2 + 6a^2c^2$ by $3ab^2 + 3a^2c^2$.
Ans. $9a^2b^4 + 27a^3b^2c^2 + 18a^4c^4$.

7. Multiply $4x^2 - 2y$ by $2y$. *Ans.* $8x^2y - 4y^2$.

8. Multiply $2x + 4y$ by $2x - 4y$. *Ans.* $4x^2 - 16y^2$.

9. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$. *Ans.* ——.

10. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

In order to bring together the similar terms, in the product o two polynomials, we arrange the terms of each polynomial witn reference to a particular letter; that is, we arrange them so tha the exponents of that letter shall go on diminishing from left to right.

11 Multiply $4a^3 - 5a^2b - 8ab^2 + 2b^3$

by $2a^2 - 3ab - 4b^2$

$$\begin{array}{l}
8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 \\
\quad - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 \\
\quad\quad - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5 \\
\hline
8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5.
\end{array}$$

After having arranged the polynomials, with reference to the letter a, multiply each term of the first, by the term $2a^2$ of the second; this gives the polynomial $8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3$, in which the signs of the terms are the same as in the multiplicand. Passing then to the term $-3ab$ of the multiplier, multiply each term of the multiplicand by it, and as it is affected with the sign $-$, affect each product with a sign contrary to that of the corresponding term in the multiplicand; this gives

$$-12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4.$$

Multiplying the multiplicand by $-4b^2$, gives

$$-16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5.$$

The product is then reduced, and we finally obtain, for the most simple expression of the product,

$$8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5.$$

12. Multiply $2a^2 - 3ax + 4x^2$ by $5a^2 - 6ax - 2x^2$.

Ans. $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$.

13. Multiply $3x^2 - 2yx + 5$ by $x^2 + 2xy - 3$.

Ans. $3x^4 + 4x^3y - 4x^2 - 4x^2y^2 + 16xy - 15$.

14. Multiply $3x^3 + 2x^2y^2 + 3y^2$ by $2x^3 - 3x^2y^2 + 5y^3$.

Ans. $\left\{\begin{array}{l} 6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 \\ -9x^2y^4 + 10x^2y^5 + 15y^5. \end{array}\right.$

15. Multiply $8ax - 6ab - c$ by $2ax + ab + c$.

Ans. $16a^2x^2 - 4a^2bx - 6a^2b^2 + 6acx - 7abc - c^2$.

16. Multiply $3a^2 - 5b^2 + 3c^2$ by $a^2 - b^3$.

Ans. $3a^4 - 5a^2b^2 + 3a^2c^2 - 3a^2b^3 + 5b^5 - 3b^3c^2$.

17. Multiply $3a^2 - 5bd + cf$
by $-5a^2 + 4bd - 8cf$.

Product $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$.

18. Multiply $4a^3b^2 - 5a^2b^2c + 8a^2bc^2 - 3a^2c^3 - 7abc^3$
by $2ab^2 - 4abc - 2bc^2 + c^3$.

Product $\left\{\begin{array}{l} 8a^4b^4 - 10a^3b^4c + 28a^3b^3c^2 - 34a^3b^2c^3 \\ -4a^2b^3c^3 - 16a^4b^3c + 12a^3bc^4 + 7a^2b^2c^4 \\ +14a^2bc^5 + 14ab^2c^5 - 3a^2c^6 - 7abc^6. \end{array}\right.$

46. REMARKS ON THE MULTIPLICATION OF POLYNOMIALS.

1st. *If both multiplicand and multiplier are homogeneous, the product will be homogeneous, and the degree of any term of the product will be indicated by the sum of the numbers which indicate the degrees of its two factors.*

Thus, in example 18th, each term of the multiplicand is of the 5th degree, and each term of the multiplier of the 3d degree: hence, each term of the product is of the 8th degree. This remark serves to discover any errors in the addition of the exponents.

2d. *If no two terms of the product are similar, there will be no reduction amongst them; and the number of terms in the product will then be equal to the number of terms in the multiplicand, multiplied by the number of terms in the multiplier.*

This is evident, since *each* term of the multiplier will produce as many terms as there are terms in the multiplicand. Thus, in example 16th, there are three terms in the multiplicand and two in the multiplier: hence, the number of terms in the product is equal to $3 \times 2 = 6$.

3d. *Among the terms of the product there are always two which cannot be reduced with any others.*

For, let us consider the product with reference to any letter common to the multiplicand and multiplier: Then the irreducible terms are,

1st. The term produced by the multiplication of the two terms of the multiplicand and multiplier which contain the highest power of this letter; and

2d. The term produced by the multiplication of the two terms which contain the lowest power of this letter.

For, these two partial products will contain this letter, to a higher and to a lower power than either of the other partial products, and consequently, they cannot be similar to any of them. This remark, the truth of which is deduced from the law of the exponents, will be very useful in division.

EXAMPLE.

$$
\begin{array}{lll}
\text{Multiply} & - \ - & 5a^4b^2 + 3a^2b - ab^4 - 2ab^3 \\
\text{by} & - \ - \ - \ - & a^2b - ab^2 \\
\hline
\text{Product} & & \left\{ \begin{array}{l} 5a^6b^3 + 3a^4b^2 - a^3b^5 - 2a^3b^4 \\ \qquad - 5a^5b^4 - 3a^3b^3 + a^2b^6 + 2a^2b^5. \end{array} \right.
\end{array}
$$

If we examine the multiplicand and multiplier, with reference to a, we see that the product of $5a^4b^2$ by a^2b, must be irreducible; also, the product of $-2ab^3$ by ab^2. If we consider the letter b, we see that the product of $-ab^4$ by $-ab^2$, must be irreducible, also that of $3a^2b$ by a^2b.

47. The following formulas depending upon the rule for multiplication, will be found useful in the practical operations of algebra.

Let a and b represent any two quantities; then $a + b$ will represent their sum, and $a - b$ their difference.

I. We have $(a + b)^2 = (a + b) \times (a + b)$,
or performing the multiplication indicated,

$$(a + b)^2 = a^2 + 2ab + b^2; \text{ that is,}$$

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

To apply this formula to finding the square of the binomial

$$5a^2 + 8a^2b,$$

we have $(5a^2 + 8a^2b)^2 = 25a^4 + 80a^4b + 64a^4b^2.$
Also, $(6a^4b + 9ab^3)^2 = 36a^8b^2 + 108a^5b^4 + 81a^2b^6.$

II. We have, $(a - b)^2 = (a - b) \times (a - b)$,
or performing the multiplication indicated,

$$(a - b)^2 = a^2 - 2ab + b^2; \text{ that is,}$$

The square of the difference between two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

To apply this to an example, we nave

$$(7a^2b^2 - 12ab^3)^2 = 49a^4b^4 - 168a^3b^5 + 144a^2b^6.$$

Also, $(4a^3b^3 \quad 7c^2d^3)^2 = 16a^6b^6 - 56a^3b^3c^2d^3 + 49c^4d^6.$

III. We have $(a+b) \times (a-b) = a^2 - b^2$, by performing the multiplication; that is,

The sum of two quantities multiplied by their difference is equal to the difference of their squares.

To apply this formula to an example, we have

$$(8a^3 + 7ab^2) \times (8a^3 - 7ab^2) = 64a^6 - 49a^2b^4.$$

48. By considering the last three results, it is perceived that their composition, or the manner in which they are formed from the multiplicand and multiplier, is entirely independent of any particular values that may be attributed to the letters a and b, which enter the two factors.

The manner in which an algebraic product is formed from its two factors, is called the *law* of the product; and this law remains always the same, whatever values may be attributed to the letters which enter into the two factors.

DIVISION.

49. Division, in algebra, is the operation for finding from two given quantities, a third quantity, which multiplied by the second shall produce the first.

The *first* quantity is called the *dividend*, the *second*, the *divisor*, and the *third*, or the quantity sought, the *quotient*.

50. It was shown in multiplication that the product of two terms having the same sign, must have the sign $+$, and that the product of two terms having unlike signs must have the sign $-$. Now, since the quotient must have such a sign that when multiplied by the divisor the product will have the sign of the dividend, we have the following rule for signs in division.

If the dividend is $+$ and the divisor $+$ the quotient is $+$;
if the dividend is $+$ and the divisor $-$ the quotient is $-$;
if the dividend is $-$ and the divisor $+$ the quotient is $-$;
if the dividend is $-$ and the divisor $-$ the quotient is $+$.

That is: *The quotient of terms having like signs is plus, and the quotient of terms having unlike signs is minus.*

51. Let us *first* consider the case in which both dividend and divisor are monomials. Take

$$35a^5b^2c^2 \text{ to be divided by } 7a^2bc;$$

The operation may be indicated thus,

$$\frac{35a^5b^2c^2}{7a^2bc}; \text{ quotient, } 5a^3bc.$$

Now, since the quotient must be such a quantity as multiplied by the divisor will produce the dividend, the co-efficient of the quotient multiplied by 7 must give 35; hence, it is 5.

Again, the exponent of each letter in the quotient must be such that when added to the exponent of the same letter in the divisor, the sum will be the exponent of that letter in the dividend. Hence, the exponent of a in the quotient is 3, the exponent of b is 1, that of c is 1, and the required quotient is $5a^3bc$.

Since we may reason in a similar manner upon any two monomials, we have for the division of monomials the following

RULE.

I. *Divide the co-efficient of the dividend by the co-efficient of the divisor, for a new co-efficient.*

II. *Write after this co-efficient, all the letters of the dividend and give to each an exponent equal to the excess of its exponent in the dividend over that in the divisor.*

By this rule we find,

$$\frac{48a^3b^3c^2d}{12ab^2c} = 4a^2bcd; \qquad \frac{150a^5b^3cd^3}{30a^3d^2} = 5a^2b^3cd.$$

EXAMPLES.

1. Divid $16x^2$ by $8x$. *Ans.* $2x$.
2. Divide $15a^2xy^3$ by $3ay$. *Ans.* $5axy^2$
3. Divide $84ab^3x$ by $12b^2$. *Ans.* $7abx$.
4. Divide $-96a^4b^2c^3$ by $12a^2bc$. *Ans.* $-8a^2bc^2$.
5. Divide $144a^9b^8c^7d^5$ by $-36a^4b^6c^6d$. *Ans.* $-4a^5b^2cd^4$.
6. Divide $-256a^3bc^2x^3$ by $-16a^2cx^2$. *Ans.* $16abcx$.
7. Divide $-300a^5b^4c^3x^2$ by $30a^4b^3c^2x$. *Ans.* $-10abcx$.
8. Divide $-400a^8b^6c^4x^5$ by $25a^8b^5c^3x$. *Ans.* $-16bcx^4$

52. It follows from the preceding rule that the exact division of monomials will be impossible:

1st. When the co-efficient of the dividend is not divisible by that of the divisor.

2d. When the exponent of the same letter is greater in the divisor than in the dividend.

This last exception includes, as we shall presently see, the case in which the divisor has a letter which is not contained in the dividend.

When either of these cases occurs, the quotient remains under the form of a monomial fraction; that is, a monomial expression, necessarily containing the algebraic sign of division. Such expressions may frequently be reduced.

Take, for example, $\dfrac{12a^4b^2cd}{8a^2bc^2} = \dfrac{3a^2bd}{2c}$.

Here, an entire monomial cannot be obtained for a quotient; for, 12 is not divisible by 8, and moreover, the exponent of c is less in the dividend than in the divisor. But the expression can be reduced, by dividing the numerator and denominator by the factors 4, a^2, b, and c, which are common to both terms of the fraction.

In general, to reduce a monomial fraction to its lowest terms:

Suppress all the factors common to both numerator and denominator.

From this rule we find,

$$\frac{48a^3b^2cd^3}{36a^2b^3c^2de} = \frac{4ad^2}{3bce}; \quad \text{and,} \quad \frac{37ab^3c^5d}{6a^3bc^4d^2} = \frac{37b^2c}{6a^2d};$$

$$\text{also,} \quad \frac{12a^8b^6c^7}{16a^7b^5c^9} = \frac{3ab}{4c^2}; \quad \text{and,} \quad \frac{7a^2b}{14a^3b^2} = \frac{1}{2ab}.$$

In the last example, as all the factors of the dividend are found in the divisor, the numerator is reduced to 1; for, in fact, both terms of the fraction are divisible by the numerator.

53. It often happens, that the exponents of certain letters are the same in the dividend and divisor.

For example, - - - - $\dfrac{24a^3b^2}{8a^2b^2}$,

is a case in which the letter b is affected with the same exponent in the dividend and divisor: hence, it will divide out, and will not appear in the quotient.

But if it is desirable to preserve the trace of this letter in the quotient, we may apply to it the rule for exponents (Art. 51), which gives

$$\frac{b^2}{b^2} = b^{2-2} = b^0.$$

The symbol b^0, indicates that the letter b enters 0 times as a factor in the quotient (Art. 16); or what is the same thing, that it does not enter it at all. Still, the notation shows that b was in the dividend and divisor with the same exponent, and has disappeared by division.

In like manner, $\dfrac{15a^2b^3c^2}{3a^2bc^2} = 5a^0b^2c^0 = 5b^2.$

54. We will now show that the power of any quantity whose exponent is 0, is equal to 1. Let the quantity be represented by a, and let m denote any exponent whatever.

Then, $\dfrac{a^m}{a^m} = a^{m-m} = a^0$, by the rule for division.

But, $\dfrac{a^m}{a^m} = 1$, since the numerator and denominator are equal:

hence, $a^0 = 1$, since each is equal to $\dfrac{a^m}{a^m}$.

We observe again, that the symbol a^0 is only employed conventionally, to preserve in the calculation the trace of a letter which entered in the enunciation of a question, but which may disappear by division.

55. In the *second* place, if the dividend is a polynomial and the divisor is a monomial, we *divide each term of the dividend by the divisor, and connect the quotients by their respective signs.*

EXAMPLES.

Divide $6a^2x^4y^6 - 12a^3x^3y^6 + 15a^4x^5y^3$ by $3a^2x^2y^2$.

Ans. $2x^2y^4 - 4axy^4 + 5a^2x^3y$.

Divide $12a^4y^6 - 16a^5y^5 + 20a^6y^4 - 28a^7y^3$ by $-4a^4y^3$.

Ans. $-3y^3 + 4ay^2 - 5a^2y + 7a^3$.

Divide $15a^2bc - 20acy^2 + 5cd^2$ by $-5abc$.

Ans. $-3a + \dfrac{4y^2}{b} - \dfrac{d^2}{ab}$.

56. In the *third* place, when both dividend and divisor are polynomials. As an example, let it be required to divide

$$26a^2b^2 + 10a^4 - 48a^3b + 24ab^3 \text{ by } 4ab - 5a^2 + 3b^2.$$

In order that we may follow the steps of the operation more easily, we will arrange the quantities with reference to the letter a.

$$\begin{array}{l|l} \textit{Dividend.} & \textit{Divisor.} \\ 10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & -5a^2 + 4ab + 3b^2 \\ \hline \end{array}$$

It follows from the definition of division and the rule for the multiplication of polynomials (Art. 45), that the dividend is the sum of the products arising from multiplying each term of the divisor by each term of the quotient sought. Hence if we could discover a term in the dividend which was derived, without reduction, from the multiplication of a term of the divisor by a term of the quotient, then, by dividing this term of the dividend by that term of the divisor, we should obtain one term of the required quotient.

Now, from the third remark of Art. 46, the term $10a^4$, containing the highest power of the letter a, is derived, without reduction from the two terms of the divisor and quotient, containing the highest power of the same letter. Hence, by dividing the term $10a^4$ by the term $-5a^2$, we shall have one term of the required quotient.

$$\begin{array}{r|l} \textit{Dividend.} & \textit{Divisor.} \\ 10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & -5a^2 + 4ab + 3b^2 \\ \cline{2-2} +10a^4 - 8a^3b - 6a^2b^2 & -2a^2 + 8ab \\ \cline{1-1} -40a^3b + 32a^2b^2 + 24ab^3 & \textit{Quotient.} \\ -40a^3b + 32a^2b^2 + 24ab^3. & \\ \cline{1-1} \end{array}$$

Since the terms $10a^4$ and $-5a^2$ are affected with contrary signs, their quotient will have the sign $-$; hence, $10a^4$, divided by $-5a^2$, gives $-2a^2$ for a term of the required quotient.

After having written this term under the divisor, multiply each term of the divisor by it, and subtract the product,

$$10a^4 - 8a^3b + 6a^2b^2,$$

from the dividend. The remainder after the first operation is

$$-40a^3b + 32a^2b^2 + 24ab^3.$$

This result is composed of the products of each term of the divisor, by all the terms of the quotient which remain to be determined. We may then consider it as a new dividend, and reason upon it as upon the proposed dividend. We will therefore divide the term $-40a^3b$, which contains the highest power of a, by the term $-5a^2$ of the divisor.

This gives $\qquad + 8ab$

for a new term of the quotient, which is written on the right of the first. Multiplying each term of the divisor by this term of the quotient, and writing the products underneath the second dividend, and making the subtraction, we find that nothing remains. Hence,

$$-2a^2 + 8ab \quad \text{or} \quad 8ab - 2a^2$$

is the required quotient, and if the divisor be multiplied by it, the product will be the given dividend.

By considering the preceding reasoning, we see that, in each operation, we divide that term of the dividend which contains the highest power of one of the letters, by that term of the divisor containing the highest power of the same letter. Now, we avoid the trouble of looking out these terms by arranging both polynomials with reference to a certain letter (Art. 45), which is then called the *leading* letter.

Since a similar course of reasoning may be had upon any two polynomials, we have for the division of polynomials the following

RULE.

I. *Arrange the dividend and divisor with reference to a certain letter, and then divide the first term on the left of the dividend by the first term on the left of the divisor, for the first term of the quotient; multiply the divisor by this term and subtract the product from the dividend.*

II. *Then divide the first term of the remainder by the first term of the divisor, for the second term of the quotient; multiply the divisor by this second term, and subtract the product from the result of the first operation. Continue the same operation until a remainder is found equal to* 0, *or till the first term of the remainder is not exactly divisible by the first term of the divisor.*

In the first case, (that is, when the remainder is 0,) the division is said to be exact. In the second case the exact division cannot be performed, and the quotient is expressed by writing the entire part obtained, and after it the remainder with its proper sign, divided by the divisor.

SECOND EXAMPLE.

Divide $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5 - 56x^5 - 18x^4y$ by $5y^2 - 8x^2 - 6xy$.

$$
\begin{array}{lrrrrrr|l}
 & -40y^5 & +68xy^4 & +25x^2y^3 & +21x^3y^2 & -18x^4y & -56x^5 & 5y^2-6xy-8x^2 \\
\cline{8-8}
 & -40y^5 & +48xy^4 & +64x^2y^3 & & & & -8y^3+4xy^2-3x^2y+7x^3 \\
\cline{2-4}
\text{1st rem.} & & 20xy^4 & -39x^2y^3 & +21x^3y^2 & & & \\
 & & 20xy^4 & -24x^2y^3 & -32x^3y^2 & & & \\
\cline{3-5}
\text{2d rem.} & & & -15x^2y^3 & +53x^3y^2 & -18x^4y & & \\
 & & & -15x^2y^3 & +18x^3y^2 & +24x^4y & & \\
\cline{4-6}
\text{3d. rem.} & & & & 35x^3y^2 & -42x^4y & -56x^5 & \\
 & & & & 35x^3y^2 & -42x^4y & -56x^5 & \\
\cline{5-7}
\text{Final remainder} & & & & & 0. & &
\end{array}
$$

57. REMARK.—In performing the division, it is not necessary to bring down all the terms of the dividend to form the first remainder, but they may be brought down in succession, as in the example.

As it is important that beginners should render themselves familiar with algebraic operations, and acquire the habit of calculating promptly, we will treat this last example in a different manner, at the same time, indicating the simplifications which should be introduced. These consist in subtracting each partial product from the dividend as soon as this product is formed.

$$
\begin{array}{l|l}
-40y^5+68xy^4+25x^2y^3+21x^3y^2-18x^4y-56x^5 & 5y^2-6xy-8x^2 \\
\hline
\text{1st rem. } \quad 20xy^4-39x^2y^3+21x^3y^2 & -8y^3+4xy^2-3x^2y+7x^3 \\
\text{2d rem. } \quad - \quad -15x^2y^3+53x^3y^2-18x^4y & \\
\text{3d rem. } \quad - \; - \; - \; - \quad 35x^3y^2-42x^4y-56x^5 & \\
\text{Final remainder } \quad - \; - \; - \quad 0. &
\end{array}
$$

First, by dividing $-40y^5$ by $5y^2$, we obtain $-8y^3$ for the quotient. Multiplying $5y^2$ by $-8y^3$, we have $-40y^5$, or, by changing the sign, $+40y^5$, which cancels the first term of the dividend.

In like manner, $-6xy \times -8y^3$ gives $+48xy^4$, or, changing the sign, $-48xy^4$, which reduced with $+68xy^4$, gives $20xy^4$ for a remainder. Again, $-8x^2 \times -8y^3$ gives $+$, and changing the sign, $-64x^2y^3$, which reduced with $25x^2y^3$, gives $-39x^2y^3$. Hence, the result of the first operation is $20xy^4-39x^2y^3$, followed by those terms of the dividend which have not been reduced with the products already obtained. For the second part of the operation, it is only necessary to bring down the next term of the dividend, to separate this new dividend from the primitive by a line, and to operate upon this new dividend in the same manner as we operated upon the primitive, and so on.

THIRD EXAMPLE.

Divide - - - $95a-73a^2+56a^4-25-59a^3$ by $-3a^2+5-11a+7a^3$.

$$
\begin{array}{l|l}
56a^4-59a^3-73a^2+95a-25 & 7a^3-3a^2-11a+5 \\
\hline
\text{1st rem. } \quad -35a^3+15a^2+55a-25 & 8a-5. \\
\text{2d remainder } \quad - \; - \quad 0. &
\end{array}
$$

GENERAL EXAMPLES.

1. Divide $10ab+15ac$ by $5a$. *Ans.* $2b+3c$.

2. Divide $30ax-54x$ by $6x$. *Ans.* $5a-9$.

3. Divide $10x^2y-15y^2-5y$ by $5y$. *Ans.* $2x^2-3y-1$.

4. Divide $12a+3ax-18ax^2$ by $3a$. *Ans.* $4+x-6x^2$.

5. Divide $6ax^2 + 9a^2x + a^2x^2$ by ax. *Ans.* $6x + 9a + ax$.

6. Divide $a^2 + 2ax + x^2$ by $a + x$. *Ans.* $a + x$.

7. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
Ans. $a^2 - 2ay + y^2$.

8. Divide $24a^2b - 12a^3cb^2 - 6ab$ by $-6ab$.
Ans. $-4a + 2a^2cb + 1$.

9. Divide $6x^4 - 96$ by $3x - 6$. *Ans.* $2x^3 + 4x^2 + 8x + 16$.

10. Divide - - $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$. *Ans.* $a^3 - 3a^2x + 3ax^2 - x^3$.

11. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
Ans. $24x^2 - 2ax - 35a^2$.

12. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$.
Ans. $y^3 + 3y^2x + 3yx^2 + x^3$.

13. Divide $64a^4b^6 - 25a^2b^8$ by $8a^2b^3 + 5ab^4$.
Ans. $8a^2b^3 - 5ab^4$.

14. Divide $6a^3 + 23a^2b + 22ab^2 + 5b^3$ by $3a^2 + 4ab + b^2$.
Ans. $2a + 5b$.

15. Divide $6ax^6 + 6ax^2y^6 + 42a^2x^2$ by $ax + 5ax$.
Ans. $x^5 + xy^6 + 7ax$.

16. Divide $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$ by $3a^2 - 5bd + cf$. *Ans.* $-5a^2 + 4bd - 8cf$.

17. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
Ans. $x^2 + xy + y^2$.

18. Divide $x^4 - y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.

19. Divide $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$.
Ans. $3a^2 - 5b^2 + 3c^2$.

20. Divide $6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 - 9x^2y^4 + 10x^2y^5 + 15y^5$ by $3x^3 + 2x^2y^2 + 3y^2$.
Ans. $2x^3 - 3x^2y^2 + 5y^3$

REMARKS ON THE DIVISION OF POLYNOMIALS.

58. The exact division of one polynomial by another is impossible:

1*st. When the first term of the arranged dividend or the first term of any of the remainders, is not exactly divisible by the first term of the arranged divisor.*

It may be added with respect to polynomials that we can often discover by mere inspection that they are not divisible. When the polynomials contain two or more letters, observe the two terms of the dividend and divisor, which contain the highest powers of each of the letters. If these terms do not give an exact quotient, we may conclude that the exact division is impossible.

Take, for example,

$$12a^3 - 5a^2b + 7ab^2 - 11b^3 \;\|\; \underline{4a^2 + 8ab + 3b^2}.$$

By considering only the letter a, the division would appear possible; but regarding the letter b, the exact division is impossible, since $-11b^3$ is not divisible by $3b^2$.

2*d. When the divisor contains a letter which is not in the dividend.*

For, it is impossible that a third quantity, multiplied by one which contains a certain letter, should give a product independent of that letter.

3*d. A monomial is never divisible by a polynomial.*

For, every polynomial multiplied by either a monomial or a polynomial gives a product containing at least two terms which are not susceptible of reduction.

4*th. If the letter, with reference to which the dividend is arranged, is not found in the divisor, the divisor is said to be independent of that letter; and in that case, the exact division is impossible, unless the divisor will divide separately the co-efficients of the different powers of the leading letter.*

For example, if the dividend were

$$3ba^4 + 9ba^2 + 12b,$$

arranged with reference to the letter a, and the divisor $3b$, the divisor would be *independent* of the letter a; and it is evident

that the exact division could not be performed unless the co-efficients of the different powers of a were exactly divisible by $3b$. The exponents of the different powers of the leading letter in the quotient would then be the same as in the dividend.

EXAMPLES.

1. Divide $18a^3x^2 - 36a^2x^3 - 12ax$ by $6x$.

Ans. $3a^3x - 6a^2x^2 - 2a$.

2. Divide $25a^4b - 30a^2b + 40ab$ by $5b$.

Ans. $5a^4 - 6a^2 + 8a$.

From the 3d remark of Art. 46, it appears that the terms of the dividend containing the highest power of the leading letter and the term containing the lowest power of the same letter are both derived, without reduction, from the multiplication of a term of the divisor by a term of the quotient. Therefore, nothing prevents our commencing the operation at the right instead of the left, since it might be performed upon the terms containing the lowest power of the letter, with reference to which the arrangement has been made.

Lastly, so independent are the partial operations required by the process, that after having subtracted the product of the divisor by the first term found in the quotient, we could obtain another term of the quotient by arranging the remainder with reference to some other letter and then proceeding as before.

If the same letter is preserved, it is only because there is no reason for changing it; and because the polynomials are already arranged with reference to it.

OF FACTORING POLYNOMIALS.

59. When a polynomial is the product of two or more factors, it is often desirable to resolve it into its component factors. This may often be done by inspection and by the aid of the formulas of Art. 47.

When one factor is a monomial, the resolution may be effected by writing the monomial for one factor, and the quotient arising

from the division of the given polynomial by this factor for the other factor.

1. Take, for example, the polynomial

$$ab + ac;$$

in which, it is plain, that a is a factor of both terms: hence

$$ab + ac = a(b + c).$$

2. Take, for a second example, the polynomial

$$ab^2c + 5ab^3 + ab^2c^2.$$

It is plain that a and b^2 are factors of all the terms: hence

$$ab^2c + 5ab^3 + ab^2c^2 = ab^2(c + 5b + c^2).$$

3. Take the polynomial $25a^4 - 30a^3b + 15a^2b^2$; it is evident that 5 and a^2 are factors of each of the terms. We may, therefore, put the polynomial under the form

$$5a^2(5a^2 - 6ab + 3b^2).$$

4. Find the factors of $3a^2b + 9a^2c + 18a^2xy$.

Ans. $3a^2(b + 3c + 6xy)$

5. Find the factors of $8a^2cx - 18acx^2 + 2ac^5y - 30a^6c^9x$.

Ans. $2ac(4ax - 9x^2 + c^4y - 15a^5c^8x)$.

6. Find the factors of $24a^2b^2cx - 30a^8b^5c^6y + 36a^7b^8cd + 6abc$.

Ans. $6abc(4abx - 5a^7b^4c^5y + 6a^6b^7d + 1)$.

By the aid of the formulas of Art. 48, polynomials having certain forms may be resolved into their binomial factors.

1. Find the factors of $a^2 + 2ab + b^2$.

Ans. $(a + b) \times (a + b)$

2. $49x^4 + 56x^3y + 16x^2y^2 = (7x^2 + 4xy)(7x^2 + 4xy)$.

3. Find the factors of $a^2 - 2ab + b^2$.

Ans. $(a - b) \times (a - b)$.

4. $64a^2b^2c^2 - 48abc^2d^2 + 9c^2d^4 = (8abc - 3cd^2)(8abc - 3cd^2)$.

5. Find the factors of $a^2 - b^2$. *Ans.* $(a + b) \times (a - b)$.

6. $16a^2c^2 - 9d^4 = (4ac + 3d^2)(4ac - 3d^2)$.

GENERAL EXAMPLES.

1. Find the factors of the polynomial $6a^3b + 8a^2b^5 - 16ab^7 - 2ab$.

2. Find the factors of the polynomial $15abc^2 - 3bc^2 + 9a^3b^5c^4 - 12db^6c^2$.

3. Find the factors of the polynomial $25a^6bc^6 - 30a^8bc^4d - 5ac^4 - 60ac^6$.

4. Find the factors of the polynomial $42a^2b^2 - 7abcd + 7abd$

Ans. $7ab(6ab - cd + d)$.

5. Find the factors of the polynomial $n^3 + 2n^2 + n$.

First, $$n^3 + 2n^2 + n = n(n^2 + 2n + 1)$$
$$= n(n+1) \times (n+1)$$
$$= n(n+1)^2.$$

6. Find the factors of the polynomial $5a^2bc + 10ab^2c + 15abc^2$.

Ans. $5abc(a + 2b + 3c)$.

7. Find the factors of the polynomial $a^2x - x^3$.

Ans. $x(a + x)(a - x)$.

60. Among the different principles of algebraic division, there is one remarkable for its applications. It is enunciated thus:

The difference of the same powers of any two quantities is exactly divisible by the difference of the quantities.

Let the quantities be represented by a and b; and let m denote any positive whole number. Then,

$$a^m - b^m$$

will express the difference between the same powers of a and b, and it is to be proved that $a^m - b^m$ is exactly divisible by $a - b$.

If we begin the division of

$$a^m - b^m \text{ by } a - b,$$

we have

$$\begin{array}{r|l} a^m - b^m & a - b \\ \hline a^m - a^{m-1}b & a^{m-1} \\ \hline \end{array}$$

1st rem. . . . . $a^{m-1}b - b^m$

or, by factoring . . . $b(a^{m-1} - b^{m-1})$.

Dividing a^m by a the quotient is a^{m-1}, by the rule for the exponents. The product of $a - b$ by a^{m-1} being subtracted from the dividend, the first remainder is $a^{m-1}b - b^m$, which can be put under the form,

$$b\,(a^{m-1} - b^{m-1}).$$

Now, if the factor

$$(a^{m-1} - b^{m-1})$$

of the remainder, be divisible by $a - b$, b times $(a^{m-1} - b^{m-1})$, must be divisible by $a - b$, and consequently $a^m - b^m$ must also be divisible by $a - b$. Hence,

If the difference of the same powers of two quantities is exactly divisible by the difference of the quantities, then, the difference of the powers of a degree greater by 1 *is also divisible by it.*

But by the rules for division, we know that $a^2 - b^2$ is divisible by $a - b$; hence, from what has just been proved, $a^3 - b^3$ must be divisible by $a - b$, and from this result we conclude that $a^4 - b^4$ is divisible by $a - b$ and so on indefinitely: hence the proposition is proved.

61. To determine the form of the quotient. If we continue the operation for division, we shall find $a^{m-2}b$ for the second term of the quotient, and $a^{m-2}b^2 - b^m$ for the second remainder; also, $a^{m-3}b^2$ for the third term of the quotient, and $a^{m-3}b^3 - b^m$ for the third remainder; and so on to the m^{th} term of the quotient, which will be

$$a^{m-m}b^{m-1} \quad \text{or} \quad b^{m-1};$$

and the m^{th} remainder will be

$$a^{m-m}b^m - b^m \quad \text{or} \quad b^m - b^m = 0.$$

Since the operation ceases when the remainder becomes 0, we shall have m terms in the quotient, and the result may be written thus:

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \ldots\ldots + ab^{m-2} + b^{m-1}.$$

CHAPTER III.

OF ALGEBRAIC FRACTIONS.

62. AN ALGEBRAIC FRACTION is an expression of one or more equal parts of 1.

One of these equal parts is called the *fractional unit.* Thus, $\frac{a}{b}$ is an algebraic fraction, and expresses that 1 has been divided into b equal parts and that a such parts are taken.

The quantity a, written above the line, is called the *numerator;* the quantity b, written below the line, the *denominator;* and both are called *terms* of the fraction.

One of the equal parts, as $\frac{1}{b}$, is called the *fractional unit;* and generally, the reciprocal of the denominator is the fractional unit.

The numerator always expresses the number of times that the fractional unit is taken; for example, in the given fraction, the fractional unit $\frac{1}{b}$ is taken a times.

63. An *entire* quantity is one which does not contain any fractional terms; thus,

$$a^2b + cx \text{ is an entire quantity.}$$

A *mixed* quantity is one which contains both entire and fractional terms; thus,

$$a^2b + \frac{c}{d} \text{ is a } \textit{mixed} \text{ quantity.}$$

Every entire quantity can be reduced to a fractional form having a given fractional unit, by multiplying it by the denominator of the fractional unit and then writing the product over the denominator; thus, the quantity c may be reduced to a fractional

form with the fractional unit $\frac{1}{b}$, by multiplying by b and dividing the product by b, which gives $\frac{bc}{b}$.

64. If the numerator is exactly divisible by the denominator, a fractional expression may be reduced to an entire one, by simply performing the division indicated; if the numerator is not exactly divisible, the application of the rule for division will sometimes reduce the fractional to a mixed quantity.

65. If the numerator a of the fraction $\frac{a}{b}$ be multiplied by any quantity, q, the resulting fraction $\frac{aq}{b}$ will express q times as many fractional units as are expressed by $\frac{a}{b}$; hence:

Multiplying the numerator of a fraction by any quantity is equivalent to multiplying the fraction by the same quantity.

66. If the denominator be multiplied by any quantity, q, the value of the fractional unit, will be diminished q times, and the resulting fraction $\frac{a}{qb}$ will express a quantity q times less than the given fraction; hence:

Multiplying the denominator of a fraction by any quantity, is equivalent to dividing the fraction by the same quantity.

67. Since we may multiply and divide an expression by the same quantity without altering its value, it follows from Arts. 65 and 66, that:

Both numerator and denominator of a fraction may be multiplied by the same quantity, without changing the value of the fraction.

In like manner it is evident that:

Both numerator and denominator of a fraction may be divided by the same quantity without changing the value of the fraction.

68. We shall now apply these principles in deducing rules for the transformation or reduction of fractions.

I. A fractional is said to be in its *simplest form* when the numerator and denominator do not contain a common factor. Now, since both terms of a fraction may be divided by the same quantity without altering its value, we have for the reduction of a fraction to its simplest form the following

RULE.

Resolve both numerator and denominator into their simple factors (Art. 59); then, suppress all the factors common to both terms, and the fraction will be in its simplest form.

REMARK.—When the terms of the fraction cannot be resolved into their simple factors by the aid of the rules already given, resort must be had to the method of the greatest common divisor, yet to be explained.

EXAMPLES.

1. Reduce the fraction $\dfrac{3ab+6ac}{3ad+12a}$ to its simplest form.

We see, by inspection, that 3 and a are factors of the numerator, hence,

$$3ab + 6ac = 3a(b + 2c).$$

We also see, that 3 and a are factors of the denominator, hence,

$$3ad + 12a = 3a(d + 4).$$

Hence, $$\frac{3ab+6ac}{3ad+12a} = \frac{3a(b+2c)}{3a(d+4)} = \frac{b+2c}{d+4}.$$

2. Reduce $\dfrac{6a^2b+3ac}{9ab+3ad}$ to its simplest form.

Ans. $\dfrac{2ab+c}{3b+d}$

3. Reduce $\dfrac{25bc+5bf}{35b^2+15b}$ to its simplest form.

Ans. $\dfrac{5c+f}{7b+3}$.

4. Reduce $\dfrac{54abc}{45a^2c+9acd}$ to its simplest form.

Ans. $\dfrac{6b}{5a+d}$

5. Reduce $\frac{36a^2b + 12abf}{84ab^2}$ to its simplest form.

Ans. $\frac{3a + f}{7b}$.

6. Reduce $\frac{12acd - 4cd^2}{12cdf + 4c^2d}$ to its simplest form.

Ans. $\frac{3a - d}{3f + c}$.

7. Reduce $\frac{18a^2c^2 - 3acf}{27ac^2 - 6ac^3}$ to its simplest form.

Ans. $\frac{6ac - f}{9c - 2c^2}$.

II. From what was shown in Art. 63, it follows that we may reduce the entire part of a mixed quantity to a fractional form with the same fractional unit as the fractional part, by multiplying and dividing it by the denominator of the fractional part. The two parts having then the same fractional unit, may be reduced by adding their numerators and writing the sum obtained over the common denominator.

Hence, to reduce a mixed quantity to a fractional form, we have the

RULE.

Multiply the entire part by the denominator of the fraction: then add the product to the numerator and write the sum over the denominator of the fractional part.

EXAMPLES.

1. Reduce $x - \frac{(a^2 - x^2)}{x}$ to the form of a fraction.

Here,
$$x - \frac{a^2 - x^2}{x} = \frac{x^2 - (a^2 - x^2)}{x} = \frac{2x^2 - a^2}{x}.$$

2. Reduce $x - \frac{ax + x^2}{2a}$ to the form of a fraction.

Ans. $\frac{ax - x^2}{2a}$.

3. Reduce $5 + \dfrac{2x - 7}{3x}$ to the form of a fraction.

Ans. $\dfrac{17x - 7}{3x}$.

4. Reduce $1 - \dfrac{x - a - 1}{a}$ to the form of a fraction.

Ans. $\dfrac{2a - x + 1}{a}$.

5. Reduce $1 + 2x - \dfrac{x - 3}{5x}$ to the form of a fraction.

Ans. $\dfrac{10x^2 + 4x + 3}{5x}$.

6. Reduce $3x - 1 - \dfrac{x + a}{3a - 2}$ to the form of a fraction.

Ans. $\dfrac{9ax - 4a - 7x + 2}{3a - 2}$.

REMARK.—We shall hereafter treat mixed quantities as though they were fractional, supposing them to have been reduced to a fractional form by the preceding rule.

III.—From Art. 64, we deduce the following rule for reducing a fractional to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator, and continue the operation so long as the first term of the remainder is divisible by the first term of the divisor: then the entire part of the quotient found, added to the quotient of the remainder by the divisor, will be the mixed quantity required.

If the remainder is 0, the division is exact, and the quotient is an entire quantity, equivalent to the given fractional expression.

EXAMPLES.

1. Reduce $\dfrac{ax + a^2}{x}$ to a mixed quantity.

Ans. $= a + \dfrac{a^2}{x}$.

2. Reduce $\dfrac{ax - x^2}{x}$ to an entire or mixed quantity.

Ans. $a - x$.

3. Reduce $\dfrac{ab - 2a^2}{b}$ to a mixed quantity.

Ans. $a - \dfrac{2a^2}{b}$.

4. Reduce $\dfrac{a^2 - x^2}{a - x}$ to an entire quantity.

Ans. $a + x$.

5. Reduce $\dfrac{x^3 - y^3}{x - y}$ to an entire quantity.

Ans. $x^2 + xy + y^2$.

6. Reduce $\dfrac{10x^2 - 5x + 3}{5x}$ to a mixed quantity.

Ans. $2x - 1 + \dfrac{3}{5x}$.

IV. To reduce fractions having different denominators to equivalent fractions having a common denominator.

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$ and $\dfrac{e}{f}$, be any three fractions whatever.

It is evident that both terms of the first fraction may be multiplied by df giving $\dfrac{adf}{bdf}$, and that this operation does not change the value of the fraction (Art. 67).

In like manner both terms of the second fraction may be multiplied by bf, giving $\dfrac{bcf}{bdf}$; also, both terms of the fraction $\dfrac{e}{f}$ may be multiplied by bd, giving $\dfrac{bde}{bdf}$.

If now we examine the three fractions $\dfrac{adf}{bdf}$, $\dfrac{bcf}{bdf}$ and $\dfrac{bde}{bdf}$, we see that they have a common denominator, bdf, and that each numerator has been obtained by multiplying the numerator of the corresponding fraction by the product of all the denominators except its own. Since we may reason in a similar manner upon any fractions whatever, we have the following

RULE.

Multiply each numerator into the product of all the denominators except its own, for new numerators, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator.

$$\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\} \text{the new numerators.}$$

and $\quad b \times c = bc \quad$ the common denominator.

2. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator. *Ans.* $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d, to equivalent fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$, to equivalent fractions having a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$ and $\frac{12a^2+24x}{12a}$.

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2+x^2}{a+x}$, to equivalent fractions having a common denominator.

Ans. $\frac{3a+3x}{6a+6x}$, $\frac{2a^3+2a^2x}{6a+6x}$ and $\frac{6a^2+6x^2}{6a+6x}$.

6. Reduce $\frac{a}{a-b}$, $\frac{c-b}{ax}$ and $\frac{b}{c}$, to equivalent fractions having a common denominator.

Ans. $\frac{a^2cx}{a^2cx-abcx}$, $\frac{ac^2-abc-bc^2+cb^2}{a^2cx-abcx}$ and $\frac{a^2bx-ab^2x}{a^2cx-abcx}$.

V. To add fractions together.

Quantities cannot be added together unless they have the same unit. Hence, the fractions must first be reduced to equivalent ones having the same fractional unit; then the sum of the numerators will designate the number of times this unit is to be taken. We have, therefore, for the addition of fractions the following

RULE.

Reduce the fractions, if necessary, to a common denominator: then add the numerators together and place their sum over the common denominator.

EXAMPLES.

1. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$.

$$\text{Here,}\quad \left.\begin{array}{l} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{array}\right\} \text{ the new numerators.}$$

$$\text{And}\quad b \times d \times f = bdf \quad \text{the common denominator.}$$

$$\text{Hence,}\quad \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf} \quad \text{the sum.}$$

2. To $a - \frac{3x^2}{b}$ add $b + \frac{2ax}{c}$. *Ans.* $a + b + \frac{2abx - 3cx^2}{bc}$.

3. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. *Ans.* $x + \frac{x}{12}$.

4. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Ans.* $\frac{19x - 14}{21}$.

5. Add $x + \frac{x-2}{3}$ to $3x + \frac{2x-3}{4}$. *Ans.* $4x + \frac{10x - 17}{12}$.

6. It is required to add $4x$, $\frac{5x^2}{2a}$ and $\frac{x+a}{2x}$ together.

Ans. $4x + \frac{5x^3 + ax + a^2}{2ax}$.

7. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$ and $\frac{2x+1}{5}$ together.

Ans. $2x + \frac{49x + 12}{60}$.

8. It is required to add $4x$, $\frac{7x}{9}$ and $2 + \frac{x}{5}$ together.

Ans. $4x + \frac{44x + 90}{45}$.

9. It is required to add $3x + \frac{2x}{5}$ and $x - \frac{8x}{9}$ together.

Ans. $3x + \frac{23x}{45}$.

10. What is the sum of $\frac{a - x}{a - b}$, $\frac{c}{a + b}$ and $\frac{d}{a + x}$.

Ans. $\frac{a^3 - ax^2 + a^2b - bx^2 + a^2c + acx - abc - bcx + a^2d - b^2d}{a^3 - b^2a + a^2x - b^2x}$

$$= \frac{a^3 + a^2(b + c + d) - a(x^2 - cx + bc) - b(x^2 + cx + bd)}{a^3 + a^2x - ab^2 - b^2x}.$$

VI. To subtract one fraction from another.

Reduce the fractional quantities to equivalent ones, having the same fractional unit; the difference of their numerators will express how many times this unit is taken in one fraction more than in the other. Hence the following

RULE.

I. *Reduce the fractions to a common denominator.*

II. *Subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.*

EXAMPLES.

1. From - - - $\frac{x - a}{2b}$ subtract $\frac{2a - 4x}{3c}$.

Here, $\left.\begin{array}{l}(x - a) \times 3c = 3cx - 3ac \\ (2a - 4x) \times 2b = 4ab - 8bx\end{array}\right\}$ the numerators.

And, $2b \times 3c = 6bc$ the common denominator

Hence, $\frac{3cx - 3ac}{6bc} - \frac{4ab - 8bx}{6bc} = \frac{3cx - 3ac - 4ab + 8bx}{6bc}$.

2. From - - $\frac{12x}{7}$ subtract $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

3. From - - $5y$ subtract $\frac{3y}{8}$. *Ans.* $\frac{37y}{8}$.

4. From - - $\frac{3x}{7}$ subtract $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

5. From - - $\frac{x+a}{b}$ subtract $\frac{c}{d}$. *Ans.* $\frac{dx+ad-bc}{bd}$.

6. From - - $\frac{3x+a}{5b}$ subtract $\frac{2x+7}{8}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$.

7. From - - $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx-ab}{bc}$.

VII. To multiply one fractional quantity by another.

Let $\frac{a}{b}$ represent any fraction, and $\frac{c}{d}$ any other fraction; and let it be required to find their product.

If, in the first place, we multiply $\frac{a}{b}$ by c, the product will be $\frac{ac}{b}$, obtained by multiplying the numerator by c, (Art. 65); but this product is d times too great, since we multiplied $\frac{a}{b}$ by a quantity d times too great. Hence, to obtain the true product we must divide by d, which is effected (Art. 66) by multiplying the denominator by d. We have then,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}; \text{ hence}$$

RULE.

I. *Cancel all factors common to the numerator and denominator.*

II. *Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

EXAMPLES.

1. Multiply $a + \frac{bx}{a}$ by $\frac{c}{d}$.

First, ... $a + \frac{bx}{a} = \frac{a^2 + bx}{a}$;

Hence, . . $\frac{a^2 + bx}{a} \times \frac{c}{d} = \frac{a^2c + bcx}{ad}$;

2. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

3. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

4. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$.

Ans. $9ax$.

5. It is required to find the product of $b + \frac{bx}{a}$ and $\frac{a}{x}$.

Ans. $\frac{ab + bx}{x}$.

6. Required the product of $\frac{x^2 - b^2}{bc}$ and $\frac{x^2 + b^2}{b + c}$.

Ans. $\frac{x^4 - b^4}{b^2c + bc^2}$.

7. Required the product of $x + \frac{x + 1}{a}$ and $\frac{x - 1}{a + b}$.

Ans. $\frac{ax^2 - ax + x^2 - 1}{a^2 + ab}$.

8. Required the product of $a + \frac{ax}{a - x}$ and $\frac{a^2 - x^2}{x + x^2}$.

Ans. $\frac{a^2 (a + x)}{x (1 + x)}$.

VIII. To divide one fraction by another.

Let $\frac{a}{b}$ represent the first, and $\frac{c}{d}$ the second fraction; then he division may be indicated thus.

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}$$

If now we multiply both numerator and denominator of this complex fraction by $\frac{d}{c}$, which will not change the value of the fraction (Art. 67), the new numerator will be $\frac{ad}{bc}$, and the new denominator $\frac{cd}{dc}$, which is equal to 1.

Hence, - - $$\frac{a}{b} \div \frac{c}{d} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{ad}{bc}\right)}{1} = \frac{ad}{bc}.$$

This last result we see might have been obtained by inverting the terms of the divisor and multiplying the dividend by the resulting fraction. Hence, for the division of fractions, we have the following

RULE.

Invert the terms of the divisor and multiply the dividend by the resulting fraction.

EXAMPLES.

1. Divide - - - $a - \frac{b}{2c}$ by $\frac{f}{g}$.

$$a - \frac{b}{2c} = \frac{2ac - b}{2c}$$

Hence, $$a - \frac{b}{2c} \div \frac{f}{g} = \frac{2ac - b}{2c} \times \frac{g}{f} = \frac{2acg - bg}{2cf}.$$

2. Let $\frac{7x}{5}$ be divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

3. Let $\dfrac{4x^2}{7}$ be divided by $5x$. *Ans.* $\dfrac{4x}{35}$.

4. Let $\dfrac{x+1}{6}$ be divided by $\dfrac{2x}{3}$. *Ans.* $\dfrac{x+1}{4x}$.

5. Let $\dfrac{x}{x-1}$ be divided by $\dfrac{x}{2}$. *Ans.* $\dfrac{2}{x-1}$.

6. Let $\dfrac{5x}{3}$ be divided by $\dfrac{2a}{3b}$. *Ans.* $\dfrac{5bx}{2a}$.

7. Let $\dfrac{x-b}{8cd}$ be divided by $\dfrac{3cx}{4d}$. *Ans.* $\dfrac{x-b}{6c^2x}$.

8. Let $\dfrac{x^4-b^4}{x^2-2bx+b^2}$ be divided by $\dfrac{x^2+bx}{x-b}$.

Ans. $x+\dfrac{b^2}{x}$.

9. Divide $\dfrac{ax-1}{1-x}$ by $\dfrac{a}{1-x^2}$. *Ans.* $\dfrac{ax(1+x)-x-1}{a}$.

10. Divide $\dfrac{a+1}{a-1}$ by $\dfrac{1+a}{1-a^2}$. *Ans.* $-(1+a)$.

69. If we have a fraction of the form

$$\frac{a}{b}=c,$$

we may observe that

$$\frac{-a}{b}=-c, \text{ also } \frac{a}{-b}=-c \text{ and } \frac{-a}{-b}=c; \text{ that is,}$$

The sign of the quotient will be changed by changing the sign either of the numerator or denominator, but will not be affected by changing the signs of both the terms.

70. We will add two propositions on the subject of fractions.

I. *If the same number be added to each of the terms of a proper fraction, the fraction resulting from these additions will be greater than the first; but if it be added to the terms of an improper fraction, the resulting fraction will be less than the first.*

Let the fraction be expressed by $\dfrac{a}{b}$.

Let m represent the number to be added to each term: then the new fraction will be, $\dfrac{a+m}{b+m}$.

In order to compare the two fractions, they must be reduced to the same denominator, which gives for

the first fraction, $$\frac{a}{b} = \frac{ab + am}{b^2 + bm}$$

and for the new fraction, $$\frac{a + m}{b + m} = \frac{ab + bm}{b^2 + bm}.$$

Now, the denominators being the same, that fraction will be the greater which has the greater numerator. But the two numerators have a common part ab, and the part bm of the second is greater than the part am of the first, when $b > a$: hence

$$ab + bm > ab + am;$$

that is, when the fraction is proper, the second fraction is greater than the first.

If the given fraction is improper, that is, if $a > b$, it is plain that the numerator of the second fraction will be less than that of the first, since bm would then be less than am.

II. *If the same number be subtracted from each term of a proper fraction, the value of the fraction will be diminished; but if it be subtracted from the terms of an improper fraction, the value of the fraction will be increased.*

Let the fraction be expressed by $\frac{a}{b}$, and denote the number to be subtracted by m.

Then, $\frac{a - m}{b - m}$ will denote the new fraction.

By reducing to the same denominator, we have,

$$\frac{a}{b} = \frac{ab - am}{b^2 - bm};$$

and $$\frac{a - m}{b - m} = \frac{ab - bm}{b^2 - bm}.$$

Now, if we suppose $a < b$, then $am < bm$; and if $am < bm$, then will

$$ab - am > ab - bm:$$

that is, the new fraction will be less than the first.

If $a > b$, that is, if the fraction is improper, then

$$am > bm, \text{ and } ab - am < ab - bm,$$

that is, the new fraction will be greater than the first.

GENERAL EXAMPLES.

1 Add $\dfrac{1+x^2}{1-x^2}$ to $\dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{2(1+x^4)}{1-x^4}$.

2. Add $\dfrac{1}{1+x}$ to $\dfrac{1}{1-x}$. *Ans.* $\dfrac{2}{1-x^2}$.

3. From $\dfrac{a+b}{a-b}$ take $\dfrac{a-b}{a+b}$. *Ans.* $\dfrac{4ab}{a^2-b^2}$.

4. From $\dfrac{1+x^2}{1-x^2}$ take $\dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{4x^2}{1-x^4}$.

5. Multiply $\dfrac{x^2-9x+20}{x^2-6x}$ by $\dfrac{x^2-13x+42}{x^2-5x}$.

Ans. $\dfrac{x^2-11x+28}{x^2}$.

6. Multiply $\dfrac{x^4-b^4}{x^2+2bx+b^2}$ by $\dfrac{x^2+bx}{x-b}$. *Ans.* x^3+b^2x.

7. Divide $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}$ by $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$.

Ans. $\dfrac{a^2+x^2}{2ax}$.

8. Divide $1+\dfrac{n-1}{n+1}$ by $1-\dfrac{n-1}{n+1}$. *Ans.* n.

EXAMPLES INDICATING USEFUL FORMS OF REDUCTION.

1. $$\frac{a}{bx}+\frac{c}{dx^2}+\frac{e}{fx^3} = \frac{adfx^5}{bdfx^6}+\frac{cbfx^4}{bdfx^6}+\frac{ebdx^3}{bdfx^6} = \frac{adfx^2+bcfx+bde}{bdfx^3}$$

2. $$\frac{a}{bx}+\frac{c}{dx^2}-\frac{e}{fx^3}-\frac{g}{hx^4} = \frac{adfhx^9}{bdfhx^{10}}+\frac{bcfhx^8}{bdfhx^{10}}-\frac{bedhx^7}{bdfhx^{10}}-\frac{bdfgx^6}{bdfhx^{10}} = \frac{adfhx^3+bcfhx^2-bedhx-bdfg}{bdfhx^4}$$

1. $$\frac{1+x^2}{1-x^2}+\frac{1-x^2}{1+x^2} = \frac{(1+x^2)^2}{(1-x^2)(1+x^2)} + \frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$= \frac{(1+x^2)^2+(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$= \frac{2(1+x^4)}{1-x^4}.$$

2. $$\frac{1}{1+x} + \frac{1}{1-x} = \frac{1-x}{(1+x)(1-x)} + \frac{1+x}{(1+x)(1-x)}$$
$$= \frac{1-x+1+x}{(1+x)(1-x)}$$
$$= \frac{2}{1-x^2}.$$

3. $$\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)^2-(a-b)^2}{(a+b)(a-b)}$$
$$= \frac{4ab}{a^2-b^2}.$$

4. $$\frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2} = \frac{(1+x^2)^2}{(1-x^2)(1+x^2)} - \frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$= \frac{(1+x^2)^2-(1-x^2)^2}{(1-x^2)(1+x^2)}$$
$$= \frac{4x^2}{1-x^4}.$$

5. $$\frac{1+x^2}{1-x^2} \div \frac{1-x^2}{1+x^2} = \frac{1+x^2}{1-x^2} \times \frac{1+x^2}{1-x^2} = \frac{(1+x^2)^2}{(1-x^2)^2}.$$

6. $$\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b} = \frac{x^4-b^4}{x^2-2bx+b^2} \times \frac{x-b}{x^2-bx}$$
$$= \frac{(x^4-b^4)(x-b)}{(x^2-2bx+b^2)(x^2+bx)}$$
$$= \frac{(x^2-b^2)(x^2+b^2)(x-b)}{(x-b)^2\ x\ (x+b)}$$
$$= \frac{(x+b)(x-b)(x^2+b^2)(x-b)}{x(x-b)(x-b)(x+b)}$$
$$= \frac{x^2+b^2}{x}.$$

Of the Symbols 0, ∞ *and* $\frac{0}{0}$.

71. The symbol 0 is called *zero*, which signifies in ordinary language, *nothing*. In Algebra, it signifies *no quantity:* it is also used to expres *a quantity less than any assignable* quantity.

The symbol ∞ is called the symbol for *infinity;* that is, it is used to represent *a quantity greater than any assignable quantity.*

If we take the fraction $\frac{a}{b}$, and suppose, whilst the value of a remains the same, that the value of b becomes greater and greater, it is evident that the value of the fraction will become less and less. When the value of b becomes very great, the value of the fraction becomes very small; and finally, when b becomes greater than any assignable quantity, or infinite, the value of the fraction becomes less than any assignable quantity, or *zero.*

Hence, we say, that a finite quantity divided by infinity is equal to zero.

We may therefore regard $\frac{a}{\infty}$, and 0, as equivalent symbols.

If in the same fraction $\frac{a}{b}$, we suppose, whilst the value of a remains the same, that the value of b becomes less and less, it is plain that the value of the fraction becomes greater and greater; and finally, when b becomes less than any assignable quantity, or *zero*, the value of the fraction becomes greater than any assignable quantity, or *infinite.*

Hence, we say, that a finite quantity divided by zero is equal to infinity.

We may then regard $\frac{a}{0}$ and ∞ as equivalent symbols: Zero and infinity are reciprocals of each other.

The expression $\frac{0}{0}$ is a symbol of *indetermination;* that is, it is employed to designate a quantity which admits of an infinite number of values. The origin of the symbol will be explained in the next chapter.

It should be observed, however, that the expression $\frac{0}{0}$ is not always a symbol of *indetermination*, but frequently arises from *the existence of a common factor*, in both terms of a fraction, which factor becomes zero, in consequence of a particular hypothesis.

1. Let us consider the value of x in the expression

$$x = \frac{a^3 - b}{a^2 - b^2}.$$

If, in this formula, a is made equal to b, there results

$$x = \frac{0}{0}.$$

But, . . . $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

and . . $a^2 - b^2 = (a - b)(a + b)$,

hence, we have,

$$x = \frac{(a - b)(a^2 + ab + b^2)}{(a - b)(a + b)}.$$

Now, if we suppress the common factor $a - b$, and then suppose $a = b$, we shall have

$$x = \frac{3a}{2}.$$

2. Let us suppose that, in another example, we have

$$x = \frac{a^2 - b^2}{(a - b)^2}.$$

If we suppose $a = b$, we have

$$x = \frac{0}{0}.$$

If, however, we suppress the factor common to the numerator and denominator, in the value of x, we have,

$$x = \frac{(a + b)(a - b)}{(a - b)(a - b)} = \frac{a + b}{a - b}.$$

If now we make $a = b$, the value of x becomes

$$\frac{2b}{0} = \infty.$$

3. Let us suppose in another example,

$$x = \frac{(a-b)^2}{a^3 - b^3},$$

in which the value of x becomes $\frac{0}{0}$ when we make $a = b$.

If we strike out the common factor $a - b$, we shall find

$$x = \frac{a - b}{a^2 + ab + b^2}.$$

If now we make $a = b$, the value of x becomes

$$\frac{0}{3a^2} = 0.$$

Therefore, before pronouncing upon the nature of the expression $\frac{0}{0}$, it is necessary to ascertain whether it does not arise from the existence of a common factor in both numerator and denominator, which becomes 0 under a particular hypothesis. If it does not arise from the existence of such a factor, we conclude that the expression is *indeterminate.* If it does arise from the existence of such a factor, strike it out, and then make the particular supposition.

If A and B represent finite quantities, the resulting value of the expression will assume one of the three forms; that is:

$$\frac{A}{B}, \quad \frac{A}{0} \quad \text{or} \quad \frac{0}{A};$$

it will be either *finite*, *infinite*, or *zero*.

This remark is of much use in the discussion of problems.

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE INVOLVING BUT ONE UNKNOWN QUANTITY

72. AN EQUATION is the algebraic expression of equality between two quantities.

Thus, $$x = a + b,$$
is an equation, and expresses that the quantity denoted by x is equal to the sum of the quantities represented by a and b.

Every equation is composed of two parts, connected by the sign of equality. The part on the left of this sign is called the *first member*, that on the right the *second member*. The second member of an equation is often 0.

73. An equation may contain *one unknown quantity* only, or it may contain *more than one*. Equations are also classified according to their degrees. *The degrees are indicated by the exponents of the unknown quantities which enter them.*

In equations involving but one unknown quantity, the degree is denoted by the exponent of the highest power of that quantity in any term.

In equations involving more than one unknown quantity, the degree is denoted by the greatest sum of the exponents of the unknown quantities in any term.

For example:

$$\left.\begin{array}{l} ax + b = cx + d \\ ax + 3by + cz + 8d = 0 \end{array}\right\}$$ are equations of the first degree.

$$\left.\begin{array}{l} ax^2 + 2bx + c = 0 \\ ax^2 + bxy + cy^2 + d = 0 \end{array}\right\}$$ are equations of the second degree.

$$\left.\begin{array}{l} c^2x^3 + 2dgx^2 = abx - c^3 \\ 4axy^2 - 2cy^3 + abxy = 3 \end{array}\right\}$$ are equations of the third degree,

and so on.

74. Equations are likewise distinguished as *numerical equations* and *literal equations.* The first are those which contain numbers only, with the exception of the unknown quantity, which is always denoted by a letter. Thus,

$$4x - 3 = 2x + 5, \qquad 3x^2 - x = 8,$$

are *numerical equations.*

A literal equation is one in which a part, or all of the known quantities, are represented by letters. Thus,

$$bx^2 + ax - 3x = 5, \quad \text{and} \quad cx + dx^2 = c + f,$$

are *literal equations.*

75. An *identical equation* is an equation in which one member is repeated in the other, or in which one member is the result of certain operations indicated in the other. In either case, the equation is true for every possible value of the unknown quantities which enter it. Thus,

$$ax + b = ax + b, \qquad (x + a)^2 = x^2 + 2ax + a^2, \qquad \frac{x^2 - y^2}{x + y} = x - y,$$

are *identical equations.*

76. From the nature of an equation, we perceive that it must possess the three following properties:

1st. The two members must be composed of quantities of the same kind.

2d. The two members must be equal to each other.

3d. The essential sign of the two members must be the same.

76.* An axiom is a self-evident proposition. We may here enumerate the following, which are employed in the *transformation* and *solution* of equations:

1. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.
2. If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.
3. If both members of an equation be multiplied by equal quantities, the products will be equal.
4. If both members of an equation be divided by equal quantities, the quotients will be equal.
5. Like powers of the two members of an equation are equal
6. Like roots of the two members of an equation are equal.

Solution of Equations of the First Degree.

77. The solution of an equation is the operation of finding a value for the unknown quantity such, that when substituted for the unknown quantity in the equation, it will satisfy it; that is, make the two members equal. This value is called a *root* of the equation.

In solving an equation, we make use of certain *transformations*.

A *transformation* of an equation is an operation by which we change its form without destroying the equality of its members.

First Transformation.

78. The object of the first transformation is, *to reduce an equation, some of whose terms are fractional, to one in which all of the terms shall be entire.*

Take the equation,

$$\frac{2x}{3} - \frac{3}{4}x + \frac{x}{6} = 11.$$

First, reduce all the fractions to the same denominator, by the known rule; the equation then becomes

$$\frac{48x}{72} - \frac{54x}{72} + \frac{12x}{72} = 11.$$

If now, both members of this equation be multiplied by 72, the equality of the members will be preserved (axiom 3), and the common denominator will disappear; and we shall have

$$48x - 54x + 12x = 792;$$ or by dividing

both members by 6, $$8x - 9x + 2x = 132.$$

The last equation could have been found in another manner by employing the least common multiple of the denominators.

The *common multiple* of two or more numbers is any number which each will divide without a remainder; and the *least* common multiple, is the least number which can be so divided.

The least common multiple of small numbers can be found by inspection. Thus, 24 is the least common multiple of 4, 6 and 8; and 12 is the least common multiple of 3, 4 and 6.

Take the last equation,

$$\frac{2x}{3} - \frac{3}{4}x + \frac{x}{6} = 11.$$

We see that 12 is the least common multiple of the denominators, and if we multiply each term of the equation by 12, reducing at the same time to entire terms, we obtain

$$8x - 9x + 2x = 132,$$

the same equation as before found.

Hence, to transform an equation involving fractional terms to one involving only entire terms, we have the following

RULE.

Form the least common multiple of all the denominators, and then multiply both members of the equation by it, reducing fractional to entire terms.

This operation is called clearing of fractions.

EXAMPLES.

1. Reduce $\frac{x}{5} + \frac{x}{4} - 3 = 20$, to an equation involving only entire terms.

We see, at once, that the least common multiple is 20, by which each term of the equation is to be multiplied.

Now, $$\frac{x}{5} \times 20 = x \times \frac{20}{5} = 4x,$$

and $$\frac{x}{4} \times 20 = x \times \frac{20}{4} = 5x;$$

that is, we reduce the fractional to entire terms, *by multiplying the numerator by the quotient of the common multiple divided by the denominator, and omitting the denominators.*

Hence, the transformed equation is

$$4x + 5x - 60 = 400.$$

2. Reduce $\frac{x}{5} + \frac{x}{7} - 4 = 3$ to an equation involving only entire terms. *Ans.* $7x + 5x - 140 = 105$.

3. Reduce $\frac{a}{b} - \frac{c}{d} + f = g$ to an equation involving only entire terms. *Ans.* $ad - bc + bdf = bdg$.

4. Reduce the equation

$$\frac{ax}{b} - \frac{2c^2x}{ab} + 4a = \frac{4bc^2x}{a^3} - \frac{5a^3}{b^2} + \frac{2c^2}{a} - 3b$$

to one involving only entire terms.

Ans. $a^4bx - 2a^2bc^2x + 4a^4b^2 = 4b^3c^2x - 5a^6 + 2a^2b^2c^2 - 3a^3b^3$.

Second Transformation.

79. The object of the second transformation is *to change any term from one member of an equation to the other.*

Let us take the equation

$$ax + b = d - cx.$$

If we add cx to both members, the equality will not be destroyed (axiom 1), and we shall have

$$ax + cx + b = d - cx + cx;$$

or by reducing, $\quad ax + cx + b = d.$

Again, if we subtract b from both members, the equality will not be destroyed (axiom 2), and we shall have, after reduction,

$$ax + cx = d - b.$$

Since we may perform similar operations on any other equation, we have, for the change or transposition of terms, the following

RULE.

Any term of an equation may be transposed from one member to the other by changing its sign.

80. We will now apply the preceding principles to the solution of equations of the first degree.

For this purpose let us assume the equation

$$\frac{a+b}{c}x - d = bx - \frac{a+d}{a}.$$

Clearing of fractions, we have,

$$a(a+b)x - acd = abcx - c(a+d).$$

If, now, we perform the operations indicated in both members, we shall obtain the equation

$$a^2x + abx - acd = abcx - ca - cd.$$

Transposing all the terms containing x, to the first member, and all the known terms to the second member, we shall have,

$$a^2x + abx - abcx = acd - ac - cd.$$

Factoring the first member, we obtain

$$(a^2 + ab - abc)\,x = acd - ac - cd:$$

If we divide both members of this equation by the co-efficient of x, we shall have

$$x = \frac{acd - ac - cd}{a^2 + ab - abc}.$$

Any other equation of the first degree may be solved in a similar manner:

Hence, in order to solve any equation of the first degree, we have the following

RULE.

I. *Clear the equation of fractions, and perform in both members all the algebraic operations indicated.*

II. *Transpose all the terms containing the unknown quantity to the first member, and all the known terms to the second member, and reduce both members to their simplest form.*

III. *Resolve the first member into two factors, one of which shall be the unknown quantity; the other one will be the algebraic sum of its several co-efficients.*

IV. *Divide both members by the co-efficient of the unknown quantity; the second member of the resulting equation will be the required value of the unknown quantity.*

1. Take the numerical example

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

Clearing of fractions

$$10x - 32x - 312 = 21 - 52x;$$

transposing and reducing

$$30x = 333:$$

Whence, by dividing both members of the equation by 30,

$$x = 11.1.$$

If we substitute this value of x, for x, in the given equation, it will verify it, that is, make the two members equal to each other.

Find the value of x in each of the following

EXAMPLES.

1. $3x - 2 + 24 = 31.$ *Ans.* $x = 3.$

2. $x + 18 = 3x - 5.$ *Ans.* $x = 11\frac{1}{2}.$

3. $6 - 2x + 10 = 20 - 3x - 2.$ *Ans.* $x = 2$

4. $x + \frac{1}{2}x + \frac{1}{3}x = 11.$ *Ans.* $x = 6.$

5. $2x - \frac{1}{2}x + 1 = 5x - 2.$ *Ans.* $x = \frac{6}{7}.$

6. $3ax + \frac{a}{2} - 3 = bx - a.$ *Ans.* $x = \frac{6 \quad - 3a}{6a - 2b}.$

7. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}.$ *Ans.* $x = 23\frac{1}{4}.$

8. $\frac{x+3}{2} + \frac{x}{3} = 4 - \frac{x-5}{4}.$ *Ans.* $x = 3\frac{6}{13}.$

9. $\frac{ax-b}{4} + \frac{a}{3} = \frac{bx}{2} - \frac{bx-a}{3}.$ *Ans.* $x = \frac{3b}{3a-2b}.$

10. $\frac{3ax}{c} - \frac{2bx}{d} - 4 = f.$ *Ans.* $x = \frac{cdf + 4cd}{3ad - 2bc}.$

11. $\frac{8ax-b}{7} - \frac{3b-c}{2} = 4 - b.$ *Ans.* $x = \frac{56 + 9b - 7c}{16a}$

12. $\frac{x}{5} - \frac{x-2}{3} + \frac{x}{2} = \frac{13}{3}.$ *Ans.* $x = 10.$

13. $\frac{x}{a} - \frac{x}{b} + \frac{x}{c} - \frac{x}{d} = f.$ *Ans.* $x = \frac{abcdf}{bcd - acd + abd - abc}.$

14. $x - \frac{3x - 5}{13} + \frac{4x - 2}{11} = x + 1.$ *Ans.* $x = 6.$

15. $\frac{x}{7} - \frac{8x}{9} - \frac{x - 3}{5} = -12\frac{29}{45}.$ *Ans.* $x = 14.$

16. $2x - \frac{4x - 2}{5} = \frac{3x - 1}{2}.$ *Ans.* $x = 3.$

17. $3x + \frac{bx - d}{3} = x + a.$ *Ans.* $x = \frac{3a + d}{6 + b}.$

18. $\frac{(a + b)(x - b)}{a - b} - 3a = \frac{4ab - b^2}{a + b} - 2x + \frac{a^2 - bx}{b}.$

Ans. $x = \frac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{2b(2a^2 + ab - b^2)}.$

Problems giving rise to Equations of the First Degree, involving but one Unknown Quantity.

81. The solution of a problem, by means of algebra, consists of two distinct parts—

1st. The statement of the problem; and

2d. The solution of the equation.

We have already explained the methods of solving the equation; and it only remains to point out the best manner of making the statement.

The *statement* of a problem is the operation of expressing, algebraically, the relations between the known and unknown quantities which enter it.

This part cannot, like the second, be subjected to any well-defined rule. Sometimes the enunciation of the problem furnishes the equation immediately; and sometimes it is necessary to discover, from the enunciation, new conditions from which an equation may be formed.

The conditions enunciated are called *explicit conditions*, and those which are deduced from them, *implicit conditions.*

In almost all cases, however, we are enabled to discover the equation by applying the following

RULE.

Denote the unknown quantity by one of the final letters of the alphabet, and then indicate, by means of algebraic signs, the same operations on the known and unknown quantities, as would be necessary to verify the value of the unknown quantity, were such value known.

PROBLEMS.

1. Find a number such, that the sum of one half, one third and one fourth of it, augmented by 45, shall be equal to 448.

Let the required number be denoted by - - - - - - x.

Then, one half of it will be denoted by - - - - - - $\frac{x}{2}$,

one third of it - - - - - - - by - - - - - - $\frac{x}{3}$,

one fourth of it - - - - - - - by - - - - - - $\frac{x}{4}$:

and by the conditions, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+45=448.$

Transposing - - $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=448-45=403;$

clearing of fractions, - - - - $6x+4x+3x=4836;$

reducing, - - - - - - - - - - $13x=4836;$

hence, - - - - - - - - - - - $x=372.$

Let us see if this value will verify the equation. We have,

$$\frac{372}{2}+\frac{372}{3}+\frac{372}{4}+45=186+124+93+45=448.$$

2. What number is that whose third part exceeds its fourth by 16?

Let the required number be denoted by x.

Then, $\frac{1}{3}x$ will denote the third part;

and $\frac{1}{4}x$ will denote the fourth part.

By the conditions of the problem,

$$\frac{1}{3}x - \frac{1}{4}x = 16.$$

Clearing of fractions, - $4x - 3x = 192;$

reducing, - - - - - $x = 192.$

Verification.

$$\frac{192}{3} - \frac{192}{4} = 16,$$

or, - - - $16 = 16.$

3. Out of a cask of wine which had leaked away a third part, 21 gallons were afterward drawn, and the cask was then half full: how much did it hold?

Suppose the cask to have held x gallons.

Then, - - - - - $\frac{x}{3}$ will denote what leaked away;

and - - - - - $\frac{x}{3} + 21$ will denote what leaked out and also what was drawn out.

By the conditions of the problem,

$$\frac{x}{3} + 21 = \frac{1}{2}x.$$

Clearing of fractions, - $2x + 126 = 3x;$

reducing - - - - - $-x = -126;$

dividing by -1 - - $x = 126.$

Verification.

$$\frac{126}{3} + 21 = \frac{126}{2};$$

or, - - - - - $63 = 63.$

4. A fish was caught whose tail weighed 9*lb.*; his head weighed as much as his tail and half his body; his body weighed as much as his head and tail together: what was the weight of the fish?

Let - - $2x$ denote the weight of the body;

then - - $9+x$ will denote weight of the head;

and since the body weighed as much as both head and tail,

$$2x = 9 + 9 + x$$

or, - $2x - x = 18$; whence, $x = 18$.

Verification.

$$2 \times 18 - 18 = 18; \quad \text{or,} \quad 18 = 18.$$

Hence, the body weighed - - - - - - - - - - - 36*lbs*,

the head weighed - - - - - - - - - - - - - 27*lbs*;

the tail weighed - - - - - - - - - - - - - 9*lbs*;

and the whole fish - - - - - - - - - - - - - 72*lbs*.

5. A person engaged a workman for 48 days. For each day that he labored he received 24 cents, and for each day that he was idle, he paid 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. *Required the number of working days, and the number of days he was idle.*

If these two numbers were known, by multiplying them respectively by 24 and 12, then subtracting the last product from the first, the result would be 504. Let us indicate these operations by means of algebraic signs.

Let - - x denote the number of working days;

then $48 - x$ will denote the number of idle days;

$24 \times x =$ the amount earned, and

$12(48 - x) =$ the amount paid for his board.

Then, from the conditions,

$$24x - 12(48 - x) = 504$$

or, $24x - 576 + 12x = 504$.

Reducing $36x = 504 + 576 = 1080$

whence, $x = 30$ the working days,

and, $48 - 30 = 18$ the idle days.

Verification.

Thirty days' labor, at 24 cents a day
amounts to - - - - - - - - - - - - - $30 \times 24 = 720$ cts;
and 18 days' board, at 12 cents a day,
amounts to - - - - - - - - - - - - $18 \times 12 = 216$ cts;
and the amount received, is their difference, 504 cts.

The preceding is but a particular case of a general problem which may be enunciated as follows.

A person engaged a workman for n days. For each day that he labored, he was to receive a cents, and for each day that he was idle, he was to pay b cents for his board. At the end of the time agreed upon, he received c cents. Required the number of working days, and the number of idle days.

Let - - x denote the number of working days; then,
$n - x$ will denote the number of idle days;
ax will denote the number of cents he received; and
$b(n - x)$ will denote the number he paid out.

From the conditions of the problem,

$$ax - b(n - x) = c.$$

Performing the indicated operations, transposing and factoring, we find,

$$(a + b)x = c + bn,$$

whence, $$x = \frac{c + bn}{a + b},$$ the number of working days; and

$$n - x = \frac{an - c}{a + b},$$ the number of idle days.

If we make $n = 48$, $a = 24$, $b = 12$ and $c = 504$, we obtain,

$$x = \frac{504 + 576}{36} = 30; \quad \text{and} \quad 48 - x = 18;$$ as before found.

6. A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6; but 3 leaps of the greyhound are equivalent to 7 of the fox. How many leaps must the greyhound make to overtake the fox?

Let us take one of the fox leaps as the unit of distance; then, 3 leaps of the greyhound being equal to 7 leaps of the fox, one of the greyhound leaps will be equal to $\frac{7}{3}$.

Let x denote the number of leaps the greyhound must make before overtaking the fox.

Then, since the fox makes 9 leaps while the hound makes 6,

$$\frac{9}{6}x \text{ or } \frac{3}{2}x$$

will denote the number of leaps the fox makes in the same time.

$\frac{7}{3}x$ will denote the whole distance passed over by the hound;

$\frac{3}{2}x$ will denote the whole distance passed over by the fox.

Then, from the conditions of the problem,

$$\frac{7}{3}x = 60 + \frac{3}{2}x.$$

Clearing of fractions, $\qquad 14x = 360 + 9x,$

transposing and reducing, $\qquad 5x = 360,$

whence, $\qquad x = 72;$

and $\qquad \frac{3}{2}x = \frac{3}{2} \times 72 = 108,$ the number of fox leaps.

Verification.

$$\frac{7 \times 72}{3} = 60 + \frac{3 \times 72}{2};$$

or, - - - - $\qquad 168 = 168.$

7. A can do a piece of work alone in 10 days, and B in 13 days: in what time can they do it if they work together?

Denote the number of days by x, and the work to be done by 1. Then, in

1 day A can do $\frac{1}{10}$ of the work; and in

1 day B can do $\frac{1}{13}$ of the work; hence, in

x days A can do $\frac{x}{10}$ of the work; and in

x days B can do $\frac{x}{13}$ of the work:

Hence, by the conditions of the question,

$$\frac{x}{10}+\frac{x}{13}=1;$$

clearing of fractions, $13x+10x=130$:
hence, $x=5\frac{15}{23}$, the number of days.

8. Divide \$1000 between A, B and C, so that A shall have \$72 more than B, and C \$100 more than A.

Ans. A's share $=$ \$324, B's $=$ \$252, C's $=$ \$424.

9. A and B play together at cards. A sits down with \$84 and B with \$48. Each loses and wins in turn, when it appears that A has five times as much as B. How much did A win? *Ans.* \$26.

10. A person dying, leaves half of his property to his wife, one sixth to each of two daughters, one twelfth to a servant, and the remaining \$600 to the poor: what was the amount of his property? *Ans.* \$7200.

11. A father leaves his property, amounting to \$2520, to four sons, A, B, C and D. C is to have \$360, B as much as C and D together, and A twice as much as B less \$1000: how much do A, B and D receive?

Ans. A \$760, B \$880, D \$520.

12. An estate of \$7500 is to be divided between a widow, two sons, and three daughters, so that each son shall receive twice as much as each daughter, and the widow herself \$500 more than all the children: what was her share, and what the share of each child?

Ans. Widow's share, \$4000.
Each son, \$1000.
Each daughter, \$500.

13. A company of 180 persons consists of men, women and children. The men are 8 more in number than the women, and the children 20 more than the men and women together: how many of each sort in the company?

Ans. 44 men, 36 women, 100 children.

14. A father divides \$2000 among five sons, so that each elder should receive \$40 more than his next younger brother: what is the share of the youngest? *Ans.* \$320.

15. A purse of \$2850 is to be divided among three persons, A, B and C; A's share is to be $\frac{6}{11}$ of B's share, and C is to have \$300 more than A and B together: what is each one's share? *Ans.* A's \$450, B's \$825, C's \$1575.

16. Two pedestrians start from the same point; the first steps twice as far as the second, but the second makes 5 steps while the first makes but one. At the end of a certain time they are 300 feet apart. Now, allowing each of the longer paces to be 3 feet, how far will each have traveled?

Ans. 1st, 200 *feet;* 2d, 500.

17. Two carpenters, 24 journeymen, and 8 apprentices, received at the end of a certain time \$144. The carpenters received \$1 per day, each journeyman half a dollar, and each apprentice 25 cents: how many days were they employed?

Ans. 9 *days.*

18. A capitalist receives a yearly income of \$2940: four fifths of his money bears an interest of 4 per cent., and the remainder of five per cent.: how much has he at interest?

Ans. \$70000.

19. A cistern containing 60 gallons of water has three unequal cocks for discharging it; the largest will empty it in one hour, the second in two hours, and the third in three: in what time will the cistern be emptied if they all run together?

Ans. $32\frac{8}{11}$ *min.*

20. In a certain orchard $\frac{1}{2}$ are apple-trees, $\frac{1}{4}$ peach-trees, $\frac{1}{6}$ plum-trees, 120 cherry-trees, and 80 pear-trees: how many trees in the orchard? *Ans.* 2400.

21. A farmer being asked how many sheep he had, answered that he had them in five fields, in the 1st he had $\frac{1}{4}$, in the 2d $\frac{1}{6}$, in the 3d $\frac{1}{8}$, in the 4th $\frac{1}{12}$, and in the 5th 450: how many had he? *Ans.* 1200.

22. My horse and saddle together are worth \$132, and the horse is worth ten times as much as the saddle: what is the value of the horse? *Ans.* \$120.

23. The rent of an estate is this year 8 per cent. greater than it was last. This year it is \$1890: what was it last year?

Ans. \$1750.

24. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

25. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and ten feet above the water: what is the whole length of the post?

Ans. 24 *feet.*

26. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse: how many guineas were in it at first?

Ans. 120.

27. A person was desirous of giving 3 pence apiece to some beggars, but found he had not money enough in his pocket by 8 pence; he therefore gave them each two pence and had 3 pence remaining: required the number of beggars. *Ans.* 11.

28. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and this done, found that he had but 12 shillings remaining: what had he at first? *Ans.* 20*s.*

29. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87, and A's money is now double B's: what did each lay out? *Ans.* \$300.

30. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where, likewise, he spent 2 shillings and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first?

Ans. 3*s.* 9*d.*

31. A farmer bought a basket of eggs, and offered them at 7 cents a dozen. But before he sold any, 5 dozen were broken by a careless boy, for which he was paid. He then sold the remainder at 8 cents a dozen, and received as much as he would have got for the whole at the first price. How many eggs had he in his basket? *Ans.* 40 *dozen.*

Equations of the First Degree involving more than one Unknown Quantity.

82. If we have an equation between two unknown quantities, we may find an *expression* for one of them in terms of the other and known quantities; but the *value* of this unknown quantity could only be determined by assuming a value for the second. Thus, from the equation,

$$x + 2y = 4,$$

we may deduce

$$x = 4 - 2y,$$

but cannot find a value for x without assuming one for y.

If, however, we have another equation between the two unknown quantities, the values of these quantities being the same in both, we may find, as before, an expression for x in terms of y, and this expression placed equal to the one already found, will give an equation containing but one unknown quantity. Let us take

$$x + 3y = 5,$$

from which we find

$$x = 5 - 3y.$$

If we place this expression equal to that before found, we deduce the equation

$$4 - 2y = 5 - 3y,$$

from the solution of which we find, $y = 1$.

This value of y, substituted in either of the given equations, gives $x = 2$: hence,

$x = 2$ and $y = 1$ satisfy both equations.

We see that in order to find determinate values for two unknown quantities, we must have two independent equations. Simultaneous equations are those in which the values of the unknown quantities are the same in them all at the same time

In the same manner it may be shown that to determine the values of three unknown quantities, we must have three equations; and generally, to determine the values of n unknown quantities we must have n equations.

Elimination.

83. *Elimination is the operation of combining several equations involving several unknown quantities, and deducing therefrom a less number of equations involving a less number of unknown quantities.*

There are three principal methods of elimination:

1st. By addition or subtraction.

2d. By substitution.

3d. By comparison.

We shall explain these methods separately.

Elimination by Addition or Subtraction.

84. Let us take the two equations

$$4x - 5y = 5,$$
$$3x + 2y = 21.$$

If we multiply both members of the first equation by 2, the co-efficient of y in the second, and both members of the second equation by 5, the co-efficient of y in the first, we obtain,

$$8x - 10y = 10,$$
$$15x + 10y = 105;$$

in which the co-efficients of y are numerically the same in both. If, now, we add these equations member to member, we find

$$23x = 115.$$

In this case y has been eliminated *by addition*.

Again, let us take the equations

$$2x + 3y = 12,$$
$$3x + 4y = 17.$$

If we multiply both members of the first equation by 3, the co-efficient of x in the second, and multiply both members of the second equation by 2, the co-efficient of x in the first, we shall have,

$$6x + 9y = 36,$$
$$6x + 8y = 34;$$

in which the co-efficients of x are the same in both. If, now, we subtract the second equation from the first, member from member, we find,

$$-y = 2.$$

Here, x has been eliminated *by subtraction.*

In a similar manner we may eliminate one unknown quantity between any two equations of the first degree containing any number of unknown quantities. The rule for elimination by addition and subtraction may be simplified by using the least common multiple. Hence, for elimination by addition or subtraction, we have the following

RULE.

Prepare the two equations in such a manner that the co-efficients of the quantity we wish to eliminate shall be numerically equal in both: then, if the two co-efficients have contrary signs, add the equations, member to member; if they have the same sign, subtract them member from member, and the resulting equation will be independent of that quantity.

Elimination by Substitution.

85. Let us take the equations,

$$5x + 7y = 43, \quad \text{and} \quad 11x + 9y = 69.$$

Find, from the first equation, the value of x in terms of y, which is,

$$x = \frac{43 - 7y}{5}.$$

Substitute this value for x in the second equation, and we shall have

$$\frac{11 \times (43 - 7y)}{5} + 9y = 69; \quad \text{or},$$

reducing, - - - $473 - 77y + 45y = 345.$

In a similar manner we may eliminate one unknown quantity between two equations of the first degree containing any number of unknown quantities.

Hence, for eliminating by substitution, we have the following

RULE.

Find from one equation the value of the unknown quantity to be eliminated in terms of the others: substitute this value in the other equation for the unknown quantity to be eliminated, and the resulting equation will be independent of that quantity.

Elimination by Comparison.

86. Let us take the equations,

$$5x + 7y = 43,$$
$$11x + 9y = 69.$$

Finding the value of x in terms of y, from both equations we have,

$$x = \frac{43 - 7y}{5},$$
$$x = \frac{69 - 9y}{11}.$$

If, now, we place these values equal to each other, we shall have,

$$\frac{43 - 7y}{5} = \frac{69 - 9y}{11};$$

reducing, - - - - $473 - 77y = 345 - 45y.$

Here, x has been eliminated. Generally, if we have two equations of the first degree containing any number of unknown quantities, any one of them may be eliminated by the following

RULE.

Find the value of the quantity we wish to eliminate, in terms of the others, from each equation, and then place these values equal to each other: the resulting equation will be independent of the quantity whose values were found.

The new equations which arise, from the two last methods of elimination, contain fractional terms. This inconvenience is avoided in the first method. The *method by substitution* is, however, advantageously employed whenever the co-efficient of either of the unknown quantities in one of the equations is equal to 1, because then the inconvenience of which we have just

spoken does not occur. We shall sometimes have occasion to employ this method, but generally the method by *addition and subtraction* is preferable. When the co-efficients are not too great, the addition or subtraction may be performed at the same time with the multiplication that is made to render the co-efficients of the same unknown quantity equal to each other.

There is also a method of elimination by means of the greatest common divisor, which will be explained in its appropriate place.

87. Let us now consider the case of three equations involving three unknown quantities.

Take the equations, $\begin{cases} 5x - 6y + 4z = 15; \\ 7x + 4y - 3z = 19, \\ 2x + y + 6z = 46. \end{cases}$

To eliminate z from the first two equations, multiply the first equation by 3 and the second by 4; and since the co-efficients of z have contrary signs, add the two results together: this gives a new equation, - - - - $43x - 2y = 121.$

Multiplying both members of the second equation by 2, a factor of the co-efficient of z in the third equation, and adding them, member to member, we have - - - $16x + 9y = 84.$

The question is then reduced to finding the values of x and y, which will satisfy these new equations.

Now, if the first be multiplied by 9, the second by 2, and the results be added together, we find

$$419x = 1257, \quad \text{whence} \quad x = 3.$$

By means of the two equations involving x and y, we may determine y as we have determined x; but the value of y may be determined more simply, since by substituting for x its value found above, the last of the two equations becomes,

$$48 + 9y = 84, \quad \text{whence} \quad y = 4.$$

In the same manner, by substituting the values of x and y, the first of the three proposed equations becomes,

$$15 - 24 + 4z = 15, \quad \text{whence} \quad z = 6.$$

If we have a group of m simultaneous equations containing m unknown quantities, it is evident, from principles already explained, that the values of these unknown quantities may be found by the following

RULE.

I. *Combine one of the m equations with each of the* $m-1$ *others, separately, eliminating the same unknown quantity; there will result* $m-1$ *equations containing* $m-1$ *unknown quantities.*

II. *Combine one of these with each of the* $m-2$ *others, separately, eliminating a second unknown quantity; there will result* $m-2$ *equations containing* $m-2$ *unknown quantities.*

III. *Continue this operation of combination and elimination till we obtain, finally, one equation containing one unknown quantity.*

IV. *Find the value of this unknown quantity by the rule for solving equations of the first degree containing one unknown quantity: substitute this value in either of the two preceding equations containing two unknown quantities, and determine the value of a second unknown quantity: substitute these two values in either of the three equations involving three unknown quantities, and so on till we find the values of them all.*

It often happens that some of the proposed equations do not contain all the unknown quantities. In this case, with a little address, the elimination is very quickly performed.

Take the four equations involving four unknown quantities,

$$2x - 3y + 2z = 13 \quad - \quad (1) \qquad 4y + 2z = 14 \quad - \quad (3).$$

$$4u - 2x = 30 \quad - \quad (2) \qquad 5y + 3u = 32 \quad - \quad (4).$$

By examining these equations, we see that the elimination of z in equations (1) and (3), will give an equation involving x and y; and if we eliminate u in the equations (2) and (4), we shall obtain a second equation, involving x and y. In the first place, the elimination of z, in (1) and (3) gives $7y - 2x = 1$ - (5), that of u, in (2) and (4), gives - - $20y + 6x = 38$ - (6).

From (5) and (6) we readily deduce the values of $y = 1$ and $x = 3$; and by substitution in (2) and (3), we also find $u = 9$ and $z = 5$.

EXAMPLES.

1. Given $2x + 3y = 16$, and $3x - 2y = 11$ to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

2. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$ to find the values of x and y. *Ans.* $x = \frac{1}{2}$, $y = \frac{1}{3}$.

3. Given $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$ to find the values of x and y. *Ans.* $x = 7$, $y = 14$.

4. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$, and $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27$ to find the values of x and y. *Ans.* $x = 60$, $y = 40$.

5. Given $\left\{\begin{array}{r} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 8$, $y = 9$, $z = 12$.

6. Given $\left\{\begin{array}{r} 2x + 4y - 3z = 22 \\ 4x - 2y + 5z = 18 \\ 6x + 7y - z = 63 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 3$, $y = 7$, $z = 4$.

7 Given $\left\{\begin{array}{r} x + \frac{1}{2}y + \frac{1}{3}z = 32 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 12$, $y = 20$, $z = 30$.

8. Given $\left\{\begin{array}{r} 7x - 2z + 3u = 17 \\ 4y - 2z + t = 11 \\ 5y - 3x - 2u = 8 \\ 4y - 3u + 2t = 9 \\ 3z + 8u = 33 \end{array}\right\}$ to find x, y, z, u, and t.

Ans. $x = 2$, $y = 4$, $z = 3$, $u = 3$, $t = 1$.

PROBLEMS GIVING RISE TO SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

1. What fraction is that, to the numerator of which, if 1 be added, its value will be one third, but if 1 be added to its denominator, its value will be one fourth?

Let x denote the numerator, and

y the denominator.

From the conditions of the problem,

$$\frac{x+1}{y} = \frac{1}{3},$$

$$\frac{x}{y+1} = \frac{1}{4}.$$

Clearing of fractions, the first equation gives,

$$3x + 3 = y,$$

and the 2d, $$4x = y + 1.$$

Whence, by eliminating y,

$$x - 3 = 1,$$

and $$x = 4.$$

Substituting, we find,

$$y = 15;$$

and the required fraction is $\frac{4}{15}$.

2. To find two numbers such that their sum shall be equal to a and their difference equal to b.

Let x denote the greater number, and

y the lesser number.

From the conditions of the problem,

$$x + y = a,$$

$$x - y = b.$$

Eliminating y by addition,

$$2x = a + b,$$

or, - - - - - - - $$x = \frac{a}{2} + \frac{b}{2}.$$

By substitution,

$$y = \frac{a}{2} - \frac{b}{2}.$$

3. A person possessed a capital of 30000 dollars, for which he drew a certain interest per annum; but he owed the sum of 20000 dollars, for which he paid a certain interest. The interest that he received exceeded that which he paid by 800 dollars. Another person possessed 35000 dollars, for which he received interest at the second of the above rates; but he owed 24000 dollars, for which he paid interest at the first of the above rates. The interest that he received exceeded that which he paid by 310 dollars. Required the two rates of interest.

Let x denote the first rate, and

y the second rate.

Then, the interest on \$30000 at x per cent. for one year will be

$$\frac{\$30000x}{100} \quad \text{or} \quad \$300x.$$

The interest on \$20000 at y per cent. for one year will be

$$\frac{\$20000y}{100} \quad \text{or} \quad \$200y.$$

Hence, from the first condition of the problem,

$$300x - 200y = 800;$$

or, - - - - $3x - 2y = 8$ - - - (1).

In like manner from the second condition of the problem we find

$$35y - 24x = 31 \quad - - - \quad (2).$$

Combining equations (1) and (2) we find,

$$y = 5 \quad \text{and} \quad x = 6.$$

Hence, the first rate is **6** per cent. and the second rate **5** per cent.

Verification.

\$30000, placed at 6 per cent., gives $\$300 \times 6 = \$1800$.
\$20000 do 5 do $\$200 \times 5 = \$1000$.
And we have $1800 - 1000 = 800$.
The second condition can be verified in the same manner.

4. There are three ingots formed by mixing together three metals in different proportions.

One pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 ounces of pewter.

One pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 ounce of pewter.

One pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 ounces of pewter.

It is required to form from these three, 1 pound of a fourth ingot which shall contain 8 ounces of silver, $3\frac{3}{4}$ ounces of copper, and $4\frac{1}{4}$ ounces of pewter.

Let x denote the number of ounces taken from the first.
y denote the number of ounces taken from the second
z denote the number of ounces taken from the third.

Now, since 1 pound or 16 ounces of the first ingot contains 7 ounces of silver, one ounce will contain $\frac{1}{16}$ of 7 ounces: that is, $\frac{7}{16}$ ounces; and

x ounces will contain $\frac{7x}{16}$ ounces of silver,

y ounces will contain $\frac{12y}{16}$ ounces of silver,

z ounces will contain $\frac{4z}{16}$ ounces of silver.

But since 1 pound of the new ingot is to contain 8 ounces of silver, we have

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8;$$

or, clearing of fractions, we have,

for the silver, $7x+12y+4z=128;$

for the copper, $3x+3y+7z=60;$

and for the pewter, $6x+y+5z=68.$

Whence, finding the values of x, y and z, we have

$x=8$, the number of ounces taken from the first.
$y=5$ " " " " " " second.
$z=3$ " " " " " " third.

5. What two numbers are they, whose sum is 33 and whose difference is 7? *Ans.* 20 and 13.

6. Divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21.

7. In a mixture of wine and cider, $\frac{1}{2}$ of the whole plus 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons, was cider; how many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

8. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; if the guinea were estimated at 21*s*., and the moidore at 27*s*., how many were there of each? *Ans.* 50.

9. Two travelers set out at the same time from London and York, whose distance apart is 150 miles; they travel toward each other; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Ans.* In 10 days.

10. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91; how many voted for each?

Ans. 233 for one, and 142 for the other.

11. A's age is double B's, and B's is triple C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's = 84, B's = 42, and C's = 14.

12. A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. £13 6*s*. 8*d*. for the horse.
£ 6 13*s*. 4*d*. for the harness.
£40 for the chaise.

13. A person has two horses, and a saddle worth £50; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first what is the value of each horse?

Ans. One £30, and the other £40.

14. Two persons, A and B, have each the same income. A saves $\frac{1}{5}$ of his yearly; but B, by spending £50 per annum more than A, at the end of 4 years finds himself £100 in debt; what is the income of each? *Ans.* £125.

15. To divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other. *Ans.* 8, 12, and 16.

16. A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery; what was its value?

Ans. £4 16*s.*

17. To divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, so obtained, will be all equal to each other.

Ans. The parts are 18, 22, 10, and 40.

18. The hour and minute hands of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 h. $5\frac{5}{11}$ min.

19. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man be in drinking it alone? *Ans.* 20 days.

20. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days; how many days would it take each person to perform the same work alone? *Ans.* A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

21. A laborer can do a certain work expressed by a, in a time expressed by b; a second laborer, the work c in a time d; a third, the work e in a time f. Required the time it would take the three laborers, working together, to perform the work g.

Ans. $\dfrac{bdfg}{adf + bcf + bde}$.

22. If 32 pounds of sea water contain 1 pound of salt, how much fresh water must be added to these 32 pounds, in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound?

Ans. 224 lbs.

23. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds; and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326.

24. A person who possessed 100000 dollars, placed the greater part of it out at 5 per cent. interest, and the other part at 4 per cent. The interest which he received for the whole amounted to 4640 dollars. Required the two parts.

Ans. \$64000 and \$36000.

25. A person possessed a certain capital, which he placed out at a certain interest. Another person possessed 10000 dollars more than the first, and putting out his capital 1 per cent. more advantageously, had an income greater by 800 dollars. A third, possessed 15000 dollars more than the first, and putting out his capital 2 per cent. more advantageously, had an income greater by 1500 dollars. Required the capitals, and the three rates of interest.

Sums at interest,	\$30000,	\$40000,	\$45000.	
Rates of interest,	4	5	6	per cent.

26. A cistern may be filled by three pipes, A, B, C. By the two first it can be filled in 70 minutes; by the first and third it can be filled in 84 minutes; and by the second and third in 140 minutes. What time will each pipe take to do it in? What time will be required, if the three pipes run together?

$$\textit{Ans.} \begin{cases} \text{A in 105 minutes.} \\ \text{B in 210 minutes.} \\ \text{C in 420 minutes.} \end{cases}$$

All will fill it in one hour.

27. A, has 3 purses, each containing a certain sum of money If \$20 be taken out of the first and put into the second, it will contain four times as much as remains in the first. If \$60 be taken from the second and put into the third, then this will contain $1\frac{3}{4}$ times as much as there remains in the second. Again, if \$40 be taken from the third and put into the first, then the third will contain $2\frac{7}{8}$ times as much as the first. What were the contents of each purse?

$$Ans. \begin{cases} \text{1st. \$120.} \\ \text{2d. \$380.} \\ \text{3d. \$500.} \end{cases}$$

28. A banker has two kinds of money; it takes a pieces of the first to make a crown, and b of the second to make the same sum. Some one offers him a crown for c pieces. How many of each kind must the banker give him?

$$Ans. \text{ 1st kind, } \frac{a(c-b)}{a-b}; \text{ 2d kind, } \frac{b(a-c)}{a-b}.$$

29. Find what each of three persons, A, B, C, is worth, knowing, 1st, that what A is worth added to l times what B and C are worth, is equal to p; 2d, that what B is worth added to m times what A and C are worth, is equal to q; 3d, that what C is worth added to n times what A and B are worth, is equal to r.

If we denote by s what A, B, and C, are worth, we introduce an auxiliary quantity, and resolve the question in a very simple manner.

30. Find the values of the estates of six persons, A, B, C, D, E, F, from the following conditions: 1st. The sum of the estates of A and B is equal to a; that of C and D is equal to b; and that of E and F is equal to c. 2d. The estate of A is worth m times that of C; the estate of D is worth n times that of E, and the estate of F is worth p times that of B.

This problem may be solved by means of a single equation, involving but one unknown quantity.

Of Indeterminate Equations and Indeterminate Problems.

88. An equation is said to be *indeterminate* when it may be satisfied for an infinite number of sets of values of the unknown quantities which enter it.

Every single equation containing two unknown quantities is indeterminate.

For example, let us take the equation

$$5x - 3y = 12,$$

whence, - - $x = \frac{12 + 3y}{5}$.

Now, by making successively,

$$y = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \&c.,$$

$$x = 3, \quad \frac{18}{5}, \quad \frac{21}{5}, \quad \frac{24}{5}, \quad \frac{27}{5}, \quad 6, \quad \&c.,$$

and any two corresponding values of x, y, being substituted in the given equation,

$$5x - 3y = 12,$$

will satisfy it: hence, there are an *infinite* number of values for x and y which will satisfy the equation, and consequently it is *indeterminate;* that is, it admits of an infinite number of solutions.

If an equation contains more than two unknown quantities, we may find an expression for one of them in terms of the others.

If, then, we assume values at pleasure for these others, we can find from this equation the corresponding values of the first; and the assumed and deduced values, taken together, will satisfy the given equation. Hence,

Every equation involving more than one unknown quantity is indeterminate.

In general, if we have n equations involving more than n unknown quantities, these equations are indeterminate; for we may, by combination and elimination, reduce them to a single equation containing more than one unknown quantity, which we have already seen is indeterminate.

If, on the contrary, we have a greater number of equations than we have unknown quantities, they cannot all be satisfied

unless some of them are dependent upon the others. If we combine them, we may eliminate all the unknown quantities, and the resulting equations, which will then contain only known quantities, will be so many *equations of condition*, which must be satisfied in order that the given equations may admit of solution.

For example, if we have

$$\begin{aligned} x + y &= a, \\ x - y &= c, \\ xy &= d; \end{aligned}$$

we may combine the first two, and find,

$$x = \frac{a}{2} + \frac{c}{2} \quad \text{and} \quad y = \frac{a}{2} - \frac{c}{2};$$

and by substituting these in the third, we shall find

$$\frac{a^2}{4} - \frac{c^2}{4} = d,$$

which expresses the relation between a, c and d, that must exist, in order that the three equations may be simultaneous.

88*. A *Problem is indeterminate* when it admits of an infinite number of solutions. This will always be the case when its enunciation involves more unknown quantities than there are given conditions; since, in that case, the statement of the problem will give rise to a less number of equations than there are unknown quantities.

1st. Let it be required to find two numbers such that 5 times the first diminished by 3 times the second shall be equal to 12.

If we denote the numbers by x and y, the conditions of the problem will give the equation

$$5x - 3y = 12,$$

which we have seen is indeterminate:—Hence, the problem admits of an infinite number of solutions, or is indeterminate.

2. Find a quantity such that if it be multiplied by a and the product increased by b, the result will be equal to c times the quantity increased by d.

Let x denote the required quantity. Then from the condition,

$$ax + b = cx + d,$$

whence, - - - - $x = \dfrac{d - b}{a - c}$.

If now we make the suppositions that $d = b$ and $a = c$, the value of x becomes $\dfrac{0}{0}$, which is a symbol of indetermination.

If we make these substitutions in the first equation, it becomes

$$ax + b = ax + b,$$

an identical equation (Art. 75), which must be satisfied for all values of x. These suppositions also render the conditions of the problem so dependent upon each other, that any quantity whatever will fulfil them all.

Hence, the result $\dfrac{0}{0}$ indicates that the problem admits of an infinite number of solutions.

3. Find two quantities such that a times the first increased by b times the second shall be equal to c, and that d times the first increased by f times the second shall be equal to g.

If we denote the quantities by x and y, we shall have from the conditions of the problem,

$$ax + by = c, \quad - - - - \quad (1)$$

$$dx + fy = g, \quad - - - - \quad (2)$$

whence - $y = \dfrac{cd - ag}{bd - af}$, and $x = \dfrac{bg - cf}{bd - af}$.

If now we make

$$cd = ag, \ (3) \qquad \text{and} \qquad af = bd, \ (4)$$

we shall find by multiplying these equations together, member by member,

$$cf = bg.$$

These suppositions, reduce the values of both x and y to $\dfrac{0}{0}$.

From (3) we find,

$$d = \frac{ag}{c}, \ \text{and from (4)} \ \ f = \frac{b}{a} \times d = \frac{bg}{c},$$

which substituted in equation (2), reduce it to

$$ax + by = c,$$

an equation which is the same as the first.

Under this supposition, we have in reality but one equation between two unknown quantities, both of which ought to be indeterminate. This supposition also renders the conditions of the problem so dependent upon each other, as to produce a less number of independent equations than there are unknown quantities.

Generally, the result $\frac{0}{0}$, with the exception of the case mentioned in Art. 71, arises from some supposition made upon the quantities entering a problem, which makes one or more conditions so dependent upon the others as to give rise to one or more indeterminate equations. In these cases the result $\frac{0}{0}$ is a true answer to the problem, and is to be interpreted as indicating that the problem admits of an infinite number of solutions.

Interpretation of Negative Results.

89. From the nature of the signs $+$ and $-$, it is clear that the operations which they indicate are diametrically opposite to each other, and it is reasonable to infer that if a positive result, that is, one affected by the sign $+$, is to be interpreted in a certain sense, that a negative result, or one affected by the sign $-$, should be interpreted in exactly the contrary sense.

To show that this inference is correct, we shall discuss one or two problems giving rise to both positive and negative results.

1. To find a number, which added to the number b, will give a sum equal to the number a.

Let x denote the required number. Then from the conditions

$$x + b = a, \quad \text{whence,} \quad x = a - b.$$

This *formula* will give the algebraic value of x in all the particular cases of the problem.

For example, let $a = 47$ and $b = 29$;
then, $x = 47 - 29 = 18$.
Again, let $a = 24$ and $b = 31$;
then, $x = 24 - 31 = -7$.

This last value of x, is called a *negative solution.* How is it to be interpreted?

If we consider it as a purely arithmetical result, that is, as arising from a series of operations in which all the quantities are regarded as positive, and in which the terms *add* and *subtract* imply, respectively, augmentation and diminution, the problem will obviously be impossible for the last values attributed to a and b; for, the number b is already greater than 24.

Considered, however, algebraically, it is not so; for we have found the value of x to be -7, and this number added, in the algebraic sense, to 31, gives 24 for the algebraic sum, and therefore satisfies both the equation and enunciation.

2. A father has lived a number of years expressed by a; his son a number of years expressed by b. Find in how many years the age of the son will be one fourth the age of the father.

Let x denote the required number of years.
Then, $a + x$ will denote the age of the father $\left.\right\}$ at the end of the
and $b + x$ will denote the age of the son $\left.\right\}$ required time.
Hence, from the conditions,

$$\frac{a + x}{4} = b + x; \quad \text{whence,} \quad x = \frac{a - 4b}{3}.$$

Suppose $a = 54$, and $b = 9$; then $x = \dfrac{54 - 36}{3} = \dfrac{18}{3} = 6$.

The father being 54 years old, and the son 9, in 6 years the father will be 60 years old, and his son 15; now 15 is the fourth of 60; hence, $x = 6$ satisfies the enunciation.

Let us now suppose $a = 45$, and $b = 15$;

then, $$x = \frac{45 - 60}{3} = -5.$$

If we substitute this value of x in the equation,

$$\frac{a+x}{4}=b+x,$$

we obtain, $$\frac{45-5}{4}=15-5;$$

or, $$10=10.$$

Hence, -5 substituted for x, verifies the equation, and therefore is a true answer.

Now, the positive result which was obtained, shows that the age of the father will be four times that of the son at the expiration of 6 years from the time when their ages were considered; while the negative result, indicates that the age of the father was four times that of his son, 5 years *previous* to the time when their ages were compared.

The question, taken in its general, or algebraic sense, demands *the time*, when the age of the father is four times that of the son. In stating it, we supposed that the time was yet to come; and so it was by the first supposition. But the conditions imposed by the second supposition, required that the time should have already passed, and the algebraic result conformed to this condition, by appearing with a negative sign.

Had we wished the result, under the second supposition, to have a positive sign, we might have altered the enunciation by demanding, *how many years since the age of the father was four times that of the son.*

If x denote the number of years, we shall have from the conditions,

$$\frac{a-x}{4}=b-x: \quad \text{hence,} \quad x=\frac{4b-a}{3}.$$

If $a=45$ and $b=15$, x will be equal to 5.

From a careful consideration of the preceding discussion, we may deduce the following principles with regard to negative results.

1st. *Every negative value found for the unknown quantity from an equation of the first degree, will, when taken with its proper sign, satisfy the equation from which it was derived.*

2d. *This negative value, taken with its proper sign, will also satisfy the conditions of the problem, understood in its algebraic sense.*

3d. *If a positive result is interpreted in a certain sense, a negative result must be interpreted in a directly contrary sense.*

4th. *The negative result, with its sign changed, may be regarded as the answer to a problem of which the enunciation only differs from the one proposed in this: that certain quantities which were additive have become subtractive, and the reverse.*

90. As a further illustration of the extent and power of the algebraic language, let us resume the general problem of the laborer, already considered.

Under the supposition that the laborer receives a sum c, we have the equations

$$\left.\begin{aligned} x + y &= n \\ ax - by &= c \end{aligned}\right\} \quad \text{whence,} \quad x = \frac{bn + c}{a + b}, \quad y = \frac{an - c}{a + b}.$$

If, at the end of the time, the laborer, instead of receiving a sum c, owed for his board a sum equal to c, then, by would be greater than ax, and under this supposition, we should have the equations,

$$x + y = n, \quad \text{and} \quad ax - by = -c.$$

Now, since the last two equations differ from the preceding two given equations only in the sign of c, if we change the sign of c, in the values of x and y, found from these equations, the results will be the values of x and y, in the last equations: this gives

$$x = \frac{bn - c}{a + b}, \quad y = \frac{an + c}{a + b}.$$

The results, for both enunciations, may be comprehended in the same formulas, by writing

$$x = \frac{bn \pm c}{a + b}, \quad y = \frac{an \mp c}{a + b}.$$

The double sign $\pm$, is read *plus* or *minus*, and $\mp$, is read, *minus* or *plus*. The upper signs correspond to the case in which the laborer received, and the lower signs, to the case in

which he owed a sum c. These formulas also comprehend the case in which, in a settlement between the laborer and his employer, their accounts balance. This supposes $c = 0$, which gives

$$x = \frac{bn}{a+b}; \quad y = \frac{an}{a+b}.$$

Discussion of Problems.

91. The *discussion* of a problem consists in making every possible supposition upon the arbitrary quantities which enter the equation of the problem, and interpreting the results.

An arbitrary quantity, is one to which we may assign a value, at pleasure.

In every general problem there is always one or more arbitrary quantities, and it is by assigning particular values to these that we get the particular cases of the general problem.

The discussion of the following problem presents nearly all the circumstances which are met with in problems giving rise to equations of the first degree.

PROBLEM OF THE COURIERS.

Two couriers are traveling along the same right line and in the same direction from R′ toward R. The number of miles traveled by one of them per hour is expressed by m, and the number of miles traveled by the other per hour, is expressed by n. Now, at a given time, say 12 o'clock, the distance between them is equal to a number of miles expressed by a: required the time when they are together.

R′ A B R.

At 12 o'clock, suppose the forward courier to be at B, the other at A, and R or R′ to be the point at which they are together.

Let a denote the distance AB, between the couriers at 12 o'clock, and suppose that distances measured to the right, from A, are regarded as positive quantities.

Let t denote the number of hours from 12 o'clock to the time when they are together.

Let x denote the distance traveled by the forward courier in t hours;

Then, $a + x$ will denote the distance traveled by the other in the same time.

Now, since the rate per hour, multiplied by the number of hours, gives the distance passed over by each, we have,

$$t \times m = a + x \quad - \ - \ - \ - \ - \quad (1)$$

$$t \times n = x \quad - \ - \ - \ - \ - \quad (2).$$

Subtracting the second equation from the first, member from member, we have,

$$t(m - n) = a;$$

whence, - - - - - $t = \dfrac{a}{m - n}$.

We will now discuss the value of t; a, m and n, being arbitrary quantities.

First, let us suppose $m > n$.

The denominator in the value of t, is then positive, and since a is a positive quantity, the value of t is also positive.

This result is interpreted as indicating that the time when they are together is *after* 12 o'clock.

The conditions of the problem confirm this interpretation. For if $m > n$, the courier from A will travel faster than the courier from B, and will therefore be continually gaining on him: the interval which separates them will diminish more and more, until it becomes 0, and then the couriers will be found upon the same point of the line.

In this case, the time t, which elapses, must be added to 12 o'clock, to obtain the time when they are together.

Second, suppose $m < n$.

The denominator, $m - n$ will then be negative, and the value of t will also be negative.

This result is interpreted in a sense exactly contrary to the interpretation of the positive result; that is, it indicates that the time of their being together was *previous* to 12 o'clock.

This interpretation is also confirmed by considering the circumstances of the problem. For, under the second supposition, the courier which is in advance travels the fastest, and therefore will continue to separate himself from the other courier. At 12 o'clock the distance between them was equal to a: after 12 o'clock it is greater than a; and as the rate of travel has not been changed, it follows that previous to 12 o'clock the distance must have been less than a. At a certain hour, therefore, before 12, the distance between them must have been equal to nothing, or the couriers were together at some point R′. The precise hour is found by *subtracting* the value of t from 12 o'clock.

Third, suppose $m = n$.

The denominator $m - n$ will then become 0, and the value of t will reduce to $\frac{a}{0}$, or ∞.

This result indicates that the length of time that must elapse before they are together is greater than any assignable time, or in other words, that they will never be together.

This interpretation is also confirmed by the conditions of the problem.

For, at 12 o'clock they are separated by a distance a, and if $m = n$ they must travel at the same rate, and we see, at once, that whatever time we allow, they can never come together; hence, the time that must elapse is *infinite*.

Fourth, suppose $a = 0$ and $m > n$ or $m < n$.

The numerator being 0, the value of the fraction is 0 or $t = 0$.

This result indicates that they are together at 12 o'clock, or that there is *no* time to be added to or subtracted from 12 o'clock.

The conditions of the problem confirm this interpretation. Because, if $a = 0$, the couriers are together at 12 o'clock; and since they travel at different rates, they could never have been together, nor can they be together after 12 o'clock: hence, t can have no other value than 0.

Fifth, suppose $a = 0$ and $m = n$.

The value of t becomes $\frac{0}{0}$, an indeterminate result.

This indicates that t may have any value whatever, or in other words, that the couriers are together at any time either before or after 12 o'clock: and this too is evident from the circumstances of the problem.

For, if $a = 0$, the couriers are together at 12 o'clock; and since they travel at the same rate, they will always be together; hence, t ought to be indeterminate.

The distances traveled by the couriers in the time t are, respectively,

$$\frac{ma}{m-n} \text{ and } \frac{na}{m-n},$$

both of which will be *plus* when $m > n$, both *minus* when $m < n$, and *infinite* when $m = n$.

In the first case t is *positive;* in the second, *negative;* and in the third, *infinite.*

When the couriers are together before 12 o'clock, the distances are negative, as they should be, since we have agreed to call distances estimated to the right *positive*, and from the rule for interpreting negative results, distances to the left ought to be regarded as *negative.*

Of Inequalities.

92. An inequality is the expression of two unequal quantities connected by the sign of inequality.

Thus, $a > b$ is an inequality, expressing that the quantity a is greater than the quantity b.

The part on the left of the sign of inequality is called the ***first member***, that on the right the *second member.*

The operations which may be performed upon equations, may in general be performed upon inequalities; but there are, nevertheless, some exceptions.

In order to be clearly understood, we will give examples of the different transformations to which inequalities may be sub

jected, taking care to point out the exceptions to which these transformations are liable.

Two inequalities are said to subsist in the same sense, when the greater quantity is in the first member in both, or in the second member in both; and in a contrary sense, when the greater quantity is in the first member of one and in the second member of the other.

Thus, $25 > 20$ and $18 > 10$, or $6 < 8$ and $7 < 9$, are inequalities which subsist in the same sense; and the in equalities

$$15 > 13 \quad \text{and} \quad 12 < 14,$$

subsist in a contrary sense.

1. *If we add the same quantity to both members of an inequality, or subtract the same quantity from both members, the resulting inequality will subsist in the same sense.*

Thus, take $8 > 6$; by adding 5, we still have

$$8 + 5 > 6 + 5;$$

and subtracting 5, we have

$$8 - 5 > 6 - 5.$$

When the two members of an inequality are both negative, that one is the least, algebraically considered, which contains the greatest number of units.

Thus, $-25 < -20$; and if 30 be added to both members, we have $5 < 10$. This must be understood entirely in an algebraic sense, and arises from the convention before established, to consider all quantities preceded by the minus sign, as subtractive.

The principle first enunciated serves to transpose certain terms from one member of the inequality to the other. Take, for ex ample, the inequality

$$a^2 + b^2 > 3b^2 - 2a^2;$$

there will result, by transposing,

$$a^2 + 2a^2 > 3b^2 - b^2, \quad \text{or} \quad 3a^2 > 2b^2.$$

2. *If two inequalities subsist in the same sense, and we add them member to member, the resulting inequality will also subsist in the same sense.*

Thus, if we add $a > b$ and $c > d$, member to member, there results $a + c > b + d$.

But this is not always the case, when we subtract, member from member, two inequalities established in the same sense.

Let there be two inequalities $4 < 7$ and $2 < 3$, we have

$$4 - 2 \text{ or } 2 < 7 - 3 \text{ or } 4.$$

But if we have the inequalities $9 < 10$ and $6 < 8$, by subtracting, we have

$$9 - 6 \text{ or } 3 > 10 - 8 \text{ or } 2.$$

We should then avoid this transformation as much as possible, or if we employ it, determine in which sense the resulting inequality subsists.

3. *If the two members of an inequality be multiplied by a positive quantity, the resulting inequality will exist in the same sense.*

Thus, - - - $a < b$, will give $3a < 3b$;
and, - - - - - $-a < -b$, $-3a < -3b$.

This principle serves to make the denominators disappear.

From the inequality $$\frac{a^2 - b^2}{2d} > \frac{c^2 - d^2}{3a},$$

we deduce, by multiplying by $6ad$,

$$3a(a^2 - b^2) > 2d(c^2 - d^2),$$

and the same principle is true for division. But,

When the two members of an inequality are multiplied or divided by a negative quantity, the resulting inequality will subsist in a contrary sense.

Take, for example, $8 > 7$; multiplying by -3, we have

$$-24 < -21.$$

In like manner, $8 > 7$ gives $\frac{8}{-3}$, or $-\frac{8}{3} < -\frac{7}{3}$.

Therefore, when the two members of an inequality are multiplied or divided by a quantity, it is necessary to ascertain whether the *multiplier* or *divisor* is negative; for, in that case, the inequality will exist in a contrary sense.

4. *It is not permitted to change the signs of the two members of an inequality, unless we establish the resulting inequality in a contrary sense;* for, this transformation is evidently the same as multiplying the two members by -1.

5. *Both members of an inequality between positive quantities can be squared, and the inequality will exist in the same sense.*

Thus, from $5 > 3$, we deduce, $25 > 9$; from $a + b > c$, we find

$$(a + b)^2 > c^2.$$

6. *When the signs of both members of the inequality are not known, we cannot tell before the operation is performed, in which sense the resulting inequality will exist.*

For example, $-2 < 3$ gives $(-2)^2$ or $4 < 9$.

But, $3 > -5$ gives, on the contrary, $(3)^2$ or $9 < (-5)^2$ or 25.

We must, then, before squaring, ascertain the signs of the two members.

Let us apply these principles to the solution of the following examples. By the solution of an inequality is meant the operation of finding an inequality, one member of which is the unknown quantity, and the other a known expression.

EXAMPLES.

1. $5x - 6 > 19.$ *Ans.* $x > 5.$

2. $3x + \frac{14}{2}x - 30 > 10.$ *Ans.* $x > 4.$

3. $\frac{x}{6} - \frac{1}{3}x + \frac{x}{2} + \frac{13}{2} > \frac{17}{2}.$ *Ans.* $x > 6.$

4. $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$ *Ans.* $x > a.$

5. $\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$ *Ans.* $x < b.$

CHAPTER V.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.— FORMATION OF THE SQUARE AND EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.—TRANSFORMATION OF RADICALS OF THE SECOND DEGREE.

93. THE *square* or second power of a number, is the product which arises from multiplying that number by itself once: for example, 49 is the square of 7, and 144 is the square of 12.

The *square root* of a number, is that number which multiplied by itself once will produce the given number. Thus, 7 is the square root of 49, and 12 the square root of 144: for, $7 \times 7 = 49$, and $12 \times 12 = 144$.

The square of a number, either entire or fractional, is easily found, being always obtained by multiplying the number by itself once. The extraction of the square root is, however, attended with some difficulty, and requires particular explanation.

The first ten numbers are,

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10,$$

and their squares,

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100.$$

Conversely, the numbers in the first line, are the *square roots* of the corresponding numbers in the second line.

We see that the square of any number, expressed by one figure, will contain no unit of a higher order than tens.

The numbers in the second line are *perfect squares*, and, generally, any number which results from multiplying a whole number by itself once, is a perfect square.

If we wish to find the square root of any number less than 100, we look in the second line, above given, and if the number is there written, the corresponding number in the first line

is its square root. If the number falls between any two numbers in the second line, its square root will fall between the corresponding numbers in the first line. Thus, 55 falls between 49 and 64; hence, its square root is greater than 7 and less than 8. Also, 91 falls between 81 and 100; hence, its square root is greater than 9 and less than 10.

If now, we change the units of the first line, 1, 2, 3, 4, &c., into units of the second order, or *tens*, by annexing 0 to each, we shall have,

10, 20, 30, 40, 50, 60, 70, 80, 90, 100,

and their corresponding squares will be,

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000:

Hence, *the square of any number of tens will contain no unit of a less denomination than hundreds.*

94. We may regard every number as composed of the sum of its tens and units.

Now, if we represent any number by N, and denote the tens by a, and the units by b, we shall have,

$$N = a + b;$$

whence, by squaring both members,

$$N^2 = a^2 + 2ab + b^2:$$

Hence, *the square of a number is equal to the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

For example, $78 = 70 + 8$, hence,

$$(78)^2 = (70)^2 + 2 \times 70 \times 8 + (8)^2 = 4900 + 1120 + 64 = 6084.$$

95. Let us now find the square root of 6084.

Since this number is expressed by more than two figures, its root will be expressed by more than one. But since it is less than 10000, which is the square of 100, the root will contain but two places of figures; that is, units and tens.

$$\dot{6}0\,\dot{8}4$$

Now, the square of the number of tens must be found in the number expressed by the two left-hand figures, which we will separate from the other two, by placing a point over the place

of units, and another over the place of hundreds. These parts, of two figures each, are called *periods.* The part 60 is comprised between the two squares 49 and 64, of which the roots are 7 and 8: hence, 7 *is the number of tens sought;* and the required root is composed of 7 tens plus a certain number of units.

The number 7 being found, we set it on the right of the given number, from which we separate it by a vertical line: then we subtract its square 49 from 60, which leaves a remainder of 11, to which we bring down the two next figures 84. The result of this operation is 1184, and this number is made up of *twice the product of the tens by the units plus the square of the units.*

$$\begin{array}{r|l} & \dot{6}0\,\dot{8}4 \;|\; 78 \\ & 49 \\ \cline{2-2} 7 \times 2 = 14\,8 & 118\,4 \\ & 118\,4 \\ \cline{2-2} & \quad\;\; 0 \end{array}$$

But since tens multiplied by units cannot give a product of a lower order than tens, it follows that the last number 4 can form no part of double the product of the tens by the units: this double product, is, therefore found in the part 118.

Now, if we double the number of tens, which gives 14, and then divide 118 by 14, the quotient 8 *is the number of units of the root*, or a greater number. This quotient can never be too small, since the part 118 will be at *least equal to twice the product of the number of tens by the units:* but it may be too large; for the 118, besides the double product of the number of tens by the units, may likewise contain tens arising from the square of the units.

To ascertain if the quotient 8 expresses the number of units, we place the 8 to the right of the 14, which gives 148, and then we multiply 148 by 8: Thus, we evidently form,

1st, the square of the units; and

2d, the double product of the tens by the units.

This multiplication being affected, gives for a product 1184, the same number as the result of the first operation. Having

subtracted the product, we find the remainder equal to 0: hence 78, is the root required.

Indeed, in the operations, we have merely subtracted from the given number 6084,

1st, the square of 7 tens or of 70;

2d, twice the product of 70 by 8; and

3d, the square of 8: that is, the three parts which enter into the composition of the square of 78.

In the same manner we may extract the square root of any number expressed by four figures.

95. Let us now extract the square root of a number expressed by more than four figures.

Let 56821444 be the number.

$$
\begin{array}{r|l}
5\dot{6}\ 8\dot{2}\ 1\dot{4}\ 4\dot{4} & \underline{7538} \\
\underline{49} & \\
1\dot{4}5 & 7\dot{8}2 \\
 & \underline{725} \\
15\dot{0}3 & 57\dot{1}4 \\
 & \underline{4509} \\
150\dot{6}8 & 12054\dot{4} \\
 & \underline{120544}
\end{array}
$$

If we consider the root as the sum of a certain number of tens and a certain number of units, the given number will, as before, be equal to the square of the tens plus twice the product of the tens by the units plus the square of the units.

If then, as before, we point off a period of two figures, at the right, the square of the tens of the required root will be found in the number 568214, at the left; and the square root of the greatest perfect square in this number will express the tens of the root.

But since this number, 568214, contains more than two figures, its root will contain more than one, (or hundreds), and the square of the hundreds will be found in the figures 5682, at the left of 14; hence, if we point off a second period 14, the square root of the greatest perfect square in 5682 will be the hundreds of the required root. But since 5682 contains more than two figures, its root will contain more than one, (or thousands), and the square of the thousands will be found in 56, at the left of 82: hence, if we point off a third period 82, the square root of the greatest perfect square in 56 will be the thousands of the required root. Hence, we place a point over 56, and then proceed thus:

Placing 7 on the right of the given number, and subtracting its square, 49, from the left hand period, we find 7 for a remainder, to which we annex the next period, 82. Separating the last figure at the right from the others by a point, and dividing the number at the left by twice 7, or 14, we have 5 for a quotient figure, which we place at the right of the figure already found, and also annex it to 14. Multiplying 145 by 5, and subtracting the product from 782, we find the remainder 57. Hence, 75 is the number of tens of tens, or hundreds, of the required square root.

To find the number of tens, bring down the next period and annex it to the second remainder, giving 5714, and divide 571 by double 75, or by 150. The quotient 3 annexed to 75 gives 753 for the number of tens in the root sought.

We may, as before, find the number of units, which in this case will be 8. Therefore, the required square root is 7538. A similar course of reasoning may be applied to a number expressed by any number of figures. Hence, for the extraction of the square root of numbers, we have the following

RULE.

I. *Separate the given number into periods of two figures each, beginning at the right hand: the period on the left will often contain but one figure.*

II. *Find the greatest perfect square in the first period on the left, and place its root on the right after the manner of a quotient in division. Subtract the square of this root from the first period, and to the remainder bring down the second period for a dividend.*

III. *Double the root already found and place it on the left for a divisor. See how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the quotien in the root and also at the right of the divisor.*

IV. *Multiply the divisor, thus augmented, by the last figure of the root found, and subtract the product from the dividend*

and to the remainder bring down the next period for a new dividend.

V. *Double the whole root already found, for a new divisor, and continue the operation as before, until all the periods are brought down.*

Remark I.—If, after all the periods are brought down, there is no remainder, the proposed number is a perfect square. But if there is a remainder, we have only found the root of the greatest perfect square contained in the given number, or *the entire part of the root sought.*

For example, if it were required to extract the square root of 168, we should find 12 for the entire part of the root and a remainder of 24, which shows that 168 is not a perfect square. But is the square of 12 the greatest perfect square contained in 168? That is, is 12 the entire part of the root?

To prove this, we will first show that, *the difference between the squares of two consecutive numbers, is equal to twice the less number augmented by* 1.

Let a represent the less number,

and $a + 1$, the greater.

Then, $(a + 1)^2 = a^2 + 2a + 1$,

and $(a)^2 = a^2$,

their difference is $\overline{\underline{\quad 2a + 1}}$ as enunciated: hence,

The entire part of the root cannot be augmented by 1, *unless the remainder is equal to, or exceeds twice the root found, plus* 1.

But, $12 \times 2 + 1 = 25$; and since the remainder 24 is less than 25, it follows that 12 cannot be augmented by a number as great as unity: hence, it is the entire part of the root.

The principle demonstrated above, may be readily applied in finding the squares of consecutive numbers.

If the numbers are large, it will be much easier to apply the above principle than to square the numbers separately.

For example, if we have $(651)^2 = 423801$, and wish to find the square of 652, we have,

$$\begin{aligned} (651)^2 &= 423801 \\ +2 \times 651 &= 1302 \\ +1 &= 1 \\ \text{and} \qquad (652)^2 &= \underline{\underline{425104}}. \\ \text{Also,} \qquad (652)^2 &= 425104 \\ +2 \times 652 &= 1304 \\ +1 &= 1 \\ \text{and} \qquad (653)^2 &= \underline{\underline{426409}}. \end{aligned}$$

REMARK II.—The number of places of figures in the root will always be equal to the number of periods into which the given number is separated.

EXAMPLES.

1. Find the square root of 7225.
2. Find the square root of 17689.
3. Find the square root of 994009.
4. Find the square root of 85678973.
5. Find the square root of 67812675.
6. Find the square root of 2792401.
7. Find the square root of 37496042.
8. Find the square root of 3661097049.
9. Find the square root of 918741672704.

REMARK III.—The square root of an imperfect square, is incommensurable with 1, that is, its value cannot be expressed in exact parts of 1.

To prove this, we shall first show that if $\frac{a}{b}$ is an irreducible fraction, its square $\frac{a^2}{b^2}$ must also be an irreducible fraction.

A number is said to be *prime* when it cannot be exactly divided by any other number, except 1. Thus 3, 5 and 7 are *prime numbers*.

It is a fundamental principle, that every number may be resolved into prime factors, and that any number thus resolved, is equal to the continued product of all its prime factors. It often happens that some of these factors are equal to each other. For example, the number

$$50 = 2 \times 5 \times 5; \quad \text{and,} \quad 180 = 2 \times 2 \times 3 \times 3 \times 5.$$

Now, from the rules for multiplication, it is evident that the square of any number is equal to the continued product of all the prime factors of that number, *each taken twice.* Hence, we see that, *the square of a number cannot contain any prime factor which is not contained in the number itself.*

But, since $\frac{a}{b}$, is, by hypothesis, an irreducible fraction, a and b can have no common factor: hence, it follows, from what has just been shown, that a^2 and b^2 cannot have a common factor, that is, $\frac{a^2}{b^2}$ is an irreducible fraction, which was to be proved.

For like reasons, $\frac{a^3}{b^3}$, $\frac{a^4}{b^4}$, - - $\frac{a^n}{b^n}$, are also irreducible fractions.

Now, let c represent any whole number which is an imperfect square. If the square root of c can be expressed by a fraction, we shall have

$$\sqrt{c} = \frac{a}{b},$$

in which $\frac{a}{b}$ is an irreducible fraction.

Squaring both members, gives,

$$c = \frac{a^2}{b^2},$$

or a whole number equal to an irreducible fraction, which is absurd; hence, $\sqrt{c}$ cannot be expressed by a fraction.

We conclude, therefore, that the square root of an imperfect square cannot be expressed in exact parts of 1. It may be shown, in a similar manner, that *any root of an imperfect power of the degree indicated, cannot be expressed in exact parts of* 1.

Extraction of the Square Root of Fractions.

96. Since the second power of a fraction is obtained by squaring the numerator and denominator separately, it follows that the square root of a fraction will be equal to the square root of the numerator divided by the square root of the denominator.

For example, $$\sqrt{\frac{a^2}{b^2}} = \frac{a}{b},$$

since $$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$$

But if the numerator and the denominator are not both perfect squares, the root of the fraction cannot be exactly found. We can, however, easily find the root to within less than the fractional unit.

Thus, if we were required to extract the square root of the fraction $\frac{a}{b}$, to within less than $\frac{1}{b}$, multiply both terms of the fractions by b, and we have $\frac{ab}{b^2}$.

Let r^2 represent the greatest perfect square in ab, then will ab be contained between r^2 and $(r+1)^2$, and $\frac{ab}{b^2}$ will be contained between

$$\frac{r^2}{b^2} \quad \text{and} \quad \frac{(r+1)^2}{b^2},$$

and the true square root of $\frac{ab}{b^2} = \frac{a}{b}$, will be contained between

$$\frac{r}{b} \quad \text{and} \quad \frac{r+1}{b};$$

but the difference between $\frac{r}{b}$ and $\frac{r+1}{b}$ is $\frac{1}{b}$; hence, either will be the square root of $\frac{a}{b}$, to within less than $\frac{1}{b}$. We have then the following

RULE.

Multiply the numerator by the denominator, and extract the square root of the product to within less than 1; divide the result by the denominator, and the quotient will be the approximate root.

For example, to extract the square root of $\frac{3}{5}$, we multiply 3 by 5, which gives 15; the perfect square nearest 15, is 16, and its square root is 4; hence, $\frac{4}{5}$ is the square root of $\frac{3}{5}$ to within less than $\frac{1}{5}$.

97. If we wish to determine the square root of a whole number which is an imperfect square, to within less than a given fractional unit, as $\frac{1}{b}$, for example, we have only to place the number under a fractional form, having the given fractional unit (Art. 63), and then we may apply the preceding rule: or what is an equivalent operation, we may

Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation; then extract the square root of the product to the nearest unit, and divide this root by the denominator of the fraction.

EXAMPLES.

1. Let it be required to extract the square root of 59, to within less than $\frac{1}{12}$.

First, $(12)^2 = 144$; and $144 \times 59 = 8496$.

Now, the square root of 8496 to the nearest unit, is 92: hence $\frac{92}{12} = 7\frac{8}{12}$, which is true to within less than $\frac{1}{12}$.

2. Find the $\sqrt{11}$ to within less than $\frac{1}{15}$. *Ans.* $3\frac{4}{15}$.

3. Find the $\sqrt{223}$ to within less than $\frac{1}{40}$. *Ans.* $14\frac{37}{40}$.

97*. The manner of determining the approximate root in decimals, is a consequence of the preceding rule.

To obtain the square root of an entire number within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., it is only necessary, according to the preceding rule, to multiply the proposed number by $(10)^2$, $(100)^2$, $(1000)^2$; or, which is the same thing,

Annex to the number, two, four, six, &c., ciphers: then extract the root of the product to the nearest unit, and divide this root by 10, 100, 1000, &c., *which is effected by pointing off one, two, three, &c., decimal places from the right hand.*

EXAMPLES.

1. To find the square root of 7 to within less than $\frac{1}{100}$.

Having multiplied by $(100)^2$, that is, having annexed four ciphers to the right hand of 7, it becomes 70000, whose root extracted to the nearest unit, is 264, which being divided by 100 gives 2.64 for the answer, and this is true to within less than $\frac{1}{100}$.

$$
\begin{array}{r|l|l}
 & 7\,\dot{0}0\,\dot{0}0 & 2.64 \\
 & 4 & \\
\hline
46 & 300 & \\
 & 276 & \\
\hline
524 & 2400 & \\
 & 2096 & \\
\hline
 & 304 \text{ Rem.} &
\end{array}
$$

2. Find the $\sqrt{29}$ to within less than $\frac{1}{100}$. *Ans.* 5.38.

3. Find the $\sqrt{227}$ to within less than $\frac{1}{10000}$. *Ans.* 15.0665.

REMARK.—The number of ciphers to be annexed to the whole number, is always double the number of decimal places required to be found in the root.

98. The manner of extracting the square root of a number containing an entire part and decimals, is deduced immediately from the preceding article.

Let us take for example the number 3.425. This is equivalent to $\frac{3425}{1000}$. Now, 1000 is not a perfect square, but the de-

nominator may be made such without altering the value of the fraction, by multiplying both terms by 10; this gives

$$\frac{34250}{10000} \text{ or } \frac{34250}{(100)^2}.$$

Then, extracting the square root of 34250 to the nearest unit, we find 185; hence, $\frac{185}{100}$ or 1.85 is the required root to within less than $\frac{1}{100}$.

If greater exactness be required, it will be necessary to annex to the number 3.425 as many ciphers as shall make the number of periods of decimals equal to the number of decimal places to be found in the root. Hence, to extract the square root of a mixed decimal:

Annex ciphers to the proposed number until the whole number of decimal places shall be equal to double the number required in the root. Then, extract the root to the nearest unit, and point off, from the right hand, the required number of decimal places.

EXAMPLES.

1. Find the $\sqrt{3271.4707}$ to within less than .01.

Ans. 57.19.

2. Find the $\sqrt{31.027}$ to within less than .01. *Ans.* 5.57.

3. Find the $\sqrt{0.01001}$ to within less than .00001.

Ans. 0.10004.

99. Finally, if it be required to find the square root of a vulgar fraction in terms of decimals:

Change the vulgar fraction into a decimal and continue the division until the number of decimal places is double the number required in the root. Then, extract the root of the decimal by the last rule.

EXAMPLE.

1. Extract the square root of $\frac{11}{14}$ to within less than .001

This number, reduced to decimals, is 0.785714 to within less than 0.000001. The root of 0.785714, to the nearest unit, is

886: hence, 0.886 is the root of $\frac{11}{14}$ to within less than 0.001.

2. Find the $\sqrt{2\frac{13}{15}}$ to within less than 0.0001. *Ans.* 1.6931.

Extraction of the Square Root of Algebraic Quantities.

100. Let us first consider the case of a monomial.

In order to discover the process for extracting the square root, let us see how the square of a monomial is formed.

By the rule for the multiplication of monomials (Art. 42), we have

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2;$$

that is, in order to square a monomial, it is necessary to *square its co-efficient, and double the exponent of each letter.*

Hence, to find the square root of a monomial,

Extract the square root of the co-efficient for a new co-efficient, and write after this, each letter, with an exponent equal to its original exponent divided by two.

Thus, $\sqrt{64a^6b^4} = 8a^3b^2$; for, $8a^3b^2 \times 8a^3b^2 = 64a^6b^4$,

and, $\sqrt{625a^2b^8c^6} = 25ab^4c^3$; for, $(25ab^4c^3)^2 = 625a^2b^8c^6$.

From the preceding rule, it follows, that, when a monomial is a perfect square, *its numerical co-efficient is a perfect square, and every exponent an even number.*

Thus, $25a^4b^2$ is a perfect square, but $98ab^4$ is not a perfect square; for, 98 is not a perfect square, and a is affected with an uneven exponent.

Of Polynomials.

101. Let us next consider the case of polynomials.

Let N denote any polynomial whatever, arranged with reference to a certain letter. Now the square of a polynomial is the product arising from multiplying the polynomial by itself once: hence, the *first term* of the product, arranged with reference to a particular letter, is the square of the first term of the polynomial, arranged with reference to the same letter.

Therefore, the square root of the first term of such a product will be the first term of the required root.

Denote this term by r, and the following terms of the root, arranged with reference to the leading letter of the polynomial, by r', r'', r''', &c., and we shall have

$$N = (r + r' + r'' + r''' + \&c.;)^2$$

or, if we designate the sum of all the terms of the root, after the first, by s,

$$N = (r + s)^2 = r^2 + 2rs + s^2$$
$$= r^2 + 2r(r' + r'' + r''' + \&c.) + s^2.$$

If now we subtract r^2 from N, and designate the remainder by R, we shall have,

$$N - r^2 = R = 2r(r' + r'' + r''' + \&c.) + s^2,$$

which remainder will evidently be arranged with reference to the leading letter of the given polynomial. If the indicated operations be performed, the first term $2rr'$ will contain a higher power of the leading letter than either of the following terms, and cannot be reduced with any of them. Hence,

If the first term of the first remainder be divided by twice the first term of the root, the quotient will be the second term of the root.

If now, we place $r + r' = n$,
and designate the sum of the remaining terms of the root, r'', r''', &c., by s', we shall have

$$N = (n + s')^2 = n^2 + 2ns' + s'^2.$$

If now we subtract n^2 from N, and denote the remainder by R', we shall have,

$$N - n^2 = R' = 2ns' + s'^2 = 2(r + r')(r'' + r''' + \&c.) + s'^2;$$

in which, if we perform the multiplications indicated in the second member, the term $2rr''$ will contain a higher power of the leading letter than either of the following terms, and cannot, consequently, be reduced with any of them. Hence,

If the first term of the second remainder be divided by twice the first term of the root, the quotient will be the third term of the root.

If we make

$$r + r' + r'' = n', \quad \text{and} \quad r''' + r^{\text{IV}} + \text{\&c.} = s'',$$

we shall have

$$N = (n' + s'')^2 = n'^2 + 2n's'' + s''^2; \text{ and}$$

$$N - n'^2 = R'' = 2(r + r' + r'')(r''' + r^{\text{IV}} + \text{\&c.}) + s''^2;$$

in which, if we perform the operations indicated, the term $2rr'''$ will contain a higher power of the leading letter than any following term. Hence,

If we divide the first term of the third remainder by twice the first term of the root, the quotient will be the fourth term of the root.

If we continue the operation, we shall see, generally, that

The first term of any remainder, divided by twice the first term of the root, will give a new term of the required root.

It should be observed, that instead of subtracting n^2 from the given polynomial, in order to find the second remainder, that that remainder could be found by subtracting $(2r + r')r'$ from the first remainder. So, the third remainder may be found by subtracting $(2n + r'')r''$ from the second, and similarly for the remainders which follow.

Hence, for the extraction of the square root of a polynomial, we have the following

RULE.

I. *Arrange the polynomial with reference to one of its letters, and then extract the square root of the first term, which will give the first term of the root. Subtract the square of this term from the given polynomial.*

II. *Divide the first term of the remainder by twice the first term of the root, and the quotient will be the second term of the root.*

III. *From the first remainder subtract the product of twice the first term of the root plus the second term, by the second term.*

IV. *Divide the first term of the second remainder by twice the first term of the root, and the quotient will be the third term of the root.*

V. *From the second remainder subtract the product of twice the sum of the first and second terms of the root, plus the third term, by the third term, and the result will be the third remainder from which the fourth term of the root may be found as before.*

VI. *Continue the operation till a remainder is found equal to* 0, *or till the first term of some remainder is not divisible by twice the first term of the root. In the former case the root found is exact, and the polynomial is a perfect square; in the latter case, it is an imperfect square.*

EXAMPLES.

1. Extract the square root of the polynomial

$$49a^2b^2 - 24ab^3 + 25a^4 - 30a^3b + 16b^4.$$

First arrange it with reference to the letter a.

$$
\begin{array}{l|l}
25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 & 5a^2 - 3ab + 4b^2 \\ \cline{2-2}
25a^4 & 10a^2 - 3ab \\ \cline{1-1}
R = -30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 & \quad\;\; -3ab \\ \cline{2-2}
\quad\;\; -30a^3b + 9a^2b^2 & -30a^3b + 9a^2b^2 \\ \hline
R' = \qquad\quad +40a^2b^2 - 24ab^3 + 16b^4 & 10a^2 - 6ab + 4b^2 \\
\qquad\qquad\;\; +40a^2b^2 - 24ab^3 + 16b^4 & \qquad\qquad\quad 4b^2 \\ \hline
R'' = \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & 40a^2b^2 - 24ab^3 + 16b^4. \\ \hline
\end{array}
$$

2. Find the square root of

$$a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

3. Find the square root of

$$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4.$$

4. Find the square root of

$$4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1.$$

5. Find the square root of

$$9a^4 - 12a^3b + 28a^2b^2 - 16ab^3 + 16b^4.$$

6. Find the square root of

$$25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2$$
$$- 36a^2bc^3 + 9a^4c^2.$$

Remarks on the Extraction of the Square Root of Polynomials.

1st. A binomial can never be a perfect square. For, its root cannot be a monomial, since the square of a monomial will be a monomial; nor can its root be a polynomial, since the square of the simplest polynomial, viz., a binomial, will contain at least three terms. Thus, an expression of the form

$$a^2 \pm b^2$$

can never be a perfect square.

2d. A trinomial, however, may be a perfect square. If so, when arranged, its two extreme terms must be squares, and the middle term double the product of the square roots of the other two. Therefore, to obtain the square root of a trinomial, when it is a perfect square,

Extract the square roots of the two extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative. To verify it, see if the double product of the two roots is equal to the middle term of the trinomial.

Thus, $9a^6 - 48a^4b^2 + 64a^2b^4$ is a perfect square,

for, $\sqrt{9a^6} = 3a^3$; and, $\sqrt{64a^2b^4} = -8ab^2$;

also, $2 \times 3a^3 \times (-8ab^2) = -48a^4b^2$, the middle term.

But $$4a^2 + 14ab + 9b^2$$

is not a perfect square: for, although $4a^2$ and $+9b^2$ are perfect squares, having for roots $2a$ and $3b$, yet $2 \times 2a \times 3b$ is not equal to $14ab$.

Of Radical Quantities of the Second Degree.

102. A *radical quantity* is the indicated root of an imperfect power of the degree indicated. Radical quantities are sometimes called *irrational* quantities, sometimes *surds*, but more commonly, simply *radicals.*

The indicated root of a perfect power of the degree indicated, is a *rational* quantity expressed under a *radical form.*

An indicated square root of an imperfect square, is called a radical of the *second degree.*

An indicated cube root of an imperfect cube, is called a radical of the *third degree.*

Generally, an indicated n^{th} root of an imperfect n^{th} power, is called a radical of the n^{th} *degree.*

Thus, $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{6}$, are radicals of the second degree;

$\sqrt[3]{4}$, $\sqrt[3]{18}$ and $\sqrt[3]{11}$, are radicals of the third degree;

and $\sqrt[n]{4}$, $\sqrt[n]{5}$ and $\sqrt[n]{11}$, are radicals of the n^{th} degree.

The degree of a radical is denoted by the index of the root.

The index of the root is also called the *index of the radical.*

103. Since like signs in both factors give a plus sign in the product, the square of $-a$, as well as that of $+a$, will be a^2: hence, the square root of a^2 is either $+a$ or $-a$. Also, the square root of $25a^2b^4$ is either $+5ab^2$ or $-5ab^2$. Whence we may conclude, that if a monomial is positive, its square root may be affected either with the sign $+$ or $-$;

thus, $$\sqrt{9a^4} = \pm 3a^2,$$

for, $+3a^2$ or $-3a^2$, squared, gives $9a^4$. The double sign $\pm$ with which the root is affected, is read *plus* or *minus.*

If the proposed monomial were *negative*, it would have no square root, since it has just been shown that the square of every quantity, whether positive or negative, is essentially positive.

Therefore, such expressions as,

$$\sqrt{-9}, \quad \sqrt{-4a^2}, \quad \sqrt{-8a^2b},$$

are algebraic symbols which indicate operations that cannot be performed. They are called *imaginary quantities*, or rather, *imaginary expressions*, and are frequently met with in the solution of equations of the second degree. Generally,

Every indicated even root of a negative quantity is an imaginary expression.

An odd root of a negative quantity may often be extracted For example, $\sqrt[3]{-27} = -3$, since $(-3)^3 = -27$.

Radicals are *similar* when they are of the same degree and the quantity under the radical sign is the same in both.

Thus, $a\sqrt{b}$ and $c\sqrt{b}$, are similar radicals of the second degree.

Of the Simplification of Radicals of the Second Degree.

104. Radicals of the second degree may often be simplified, and otherwise transformed, by the aid of the following principles.

1st. Let the $\sqrt{a}$, and $\sqrt{b}$, denote any two radicals of the second degree, and denote their product by p; whence,

$$\sqrt{a}\times\sqrt{b}=p \quad - \; - \; - \; - \quad (1).$$

Squaring both members of equation (1), (axiom 5), we have,

$$(\sqrt{a})^2\times(\sqrt{b})^2=p^2,$$

$$\text{or,} \; - \; - \; - \; - \; - \; - \; - \quad ab=p^2 \quad - \; - \; - \; - \quad (2).$$

Extracting the square root of both members of equation (2), (axiom 6), we have,

$$\sqrt{ab}=p\,;$$

but things which are equal to the same thing are equal to each other, whence,

$$\sqrt{a}\times\sqrt{b}=\sqrt{ab}\,;\ \text{hence,}$$

The product of the square roots of two quantities is equal to the square root of the product of those quantities.

2d. Denote the quotient of $\sqrt{a}$ by $\sqrt{b}$, by q; whence,

$$\frac{\sqrt{a}}{\sqrt{b}}=q \quad - \; - \; - \; - \quad (1).$$

Squaring both members of equation (1), we find,

$$\frac{(\sqrt{a})^2}{(\sqrt{b})^2}=q^2,$$

$$\text{or,} \qquad \frac{a}{b}=q^2 \quad - \; - \; - \; - \quad (2).$$

Extracting the square root of both members of equation (2), we have,

$$\sqrt{\frac{a}{b}}=q.$$

Things which are equal to the same thing are equal to each other, whence,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}; \quad \text{hence,}$$

The quotient of the square roots of two quantities is equal to the square root of the quotient of the same quantities.

105. The square root of $98ab^4$ may be placed under the form

$$\sqrt{98ab^4} = \sqrt{49b^4 \times 2a},$$

which, from the 1st principle above, may be written,

$$\sqrt{49b^4} \times \sqrt{2a} = 7b^2\sqrt{2a}.$$

In like manner,

$$\sqrt{45a^2b^3c^2d} = \sqrt{9a^2b^2c^2 \times 5bd} = 3abc\sqrt{5bd}.$$

$$\sqrt{864a^2b^5c^{11}} = \sqrt{144a^2b^4c^{10} \times 6bc} = 12ab^2c^5\sqrt{6bc}.$$

The quantity which stands without the radical sign is called the *co-efficient* of the radical.

Thus, $7b^2$, $3abc$, and $12ab^2c^5$, are *co-efficients of the radicals.*

In general, to simplify a radical of the second degree:

I. *Resolve the quantity under the radical sign into two factors, one of which shall be the greatest perfect square which enters it as a factor.*

II. *Write the square root of the perfect square before the radical sign, under which place the other factor.*

EXAMPLES.

1. Reduce $\sqrt{75a^3bc}$ to its simplest form.

2. Reduce $\sqrt{128b^5a^6d^2}$ to its simplest form.

3. Reduce $\sqrt{32a^9b^8c}$ to its simplest form.

4. Reduce $\sqrt{256a^2b^4c^8}$ to its simplest form.

5. Reduce $\sqrt{1024a^9b^7c^5}$ to its simplest form.

6. Reduce $\sqrt{728a^7b^5c^6d}$ to its simplest form.

If the quantity under the radical sign is a polynomial, we may often simplify the expression by the same rule.

Take, for example, the expression

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3}.$$

The quantity under the radical sign is not a perfect square: but it can be put under the form

$$ab\,(a^2 + 4ab + 4b^2).$$

Now, the factor within the parenthesis is evidently the square of $a + 2b$, whence we have

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3} = (a + 2b)\sqrt{ab}.$$

105*. Conversely, we may introduce a factor under the radical sign.

Thus, $$a\sqrt{b} = \sqrt{a^2}\sqrt{b},$$

which by article 104, is equal to $\sqrt{a^2b}$. Hence,

The co-efficient of a radical may be passed under the radical sign, as a factor, by squaring it.

The principal use of this transformation, is to find an approximate value of any radical, which shall differ from its true value, by less than 1.

For example, take the expression $6\sqrt{13}$.

Now, as 13 is not a perfect square, we can only find an approximate value for its square root; and when this approximate value is multiplied by 6, the product will differ materially from the true value of $6\sqrt{13}$. But if we write,

$$6\sqrt{13} = \sqrt{6^2 \times 13} = \sqrt{36 \times 13} = \sqrt{468},$$

we find that the square root of 468 is the whole number 21, to within less than 1. Hence,

$$6\sqrt{13} = 21, \text{ to within less than 1.}$$

In a similar manner we may find,

$$12\sqrt{7} = 31, \text{ to within less than 1.}$$

Addition and Subtraction.

106. In order to add or substract similar radicals:

Add or subtract their co-efficients, and to the sum or difference annex the common radical.

Thus, $\qquad 3a\sqrt{b} + 5c\sqrt{b} = (3a + 5c)\sqrt{b}$;

and $\qquad 3a\sqrt{b} - 5c\sqrt{b} = (3a - 5c)\sqrt{b}$.

In like manner,

$$7\sqrt{2a} + 3\sqrt{2a} = (7 + 3)\sqrt{2a} = 10\sqrt{2a};$$

and $$7\sqrt{2a} - 3\sqrt{2a} = (7 - 3)\sqrt{2a} = 4\sqrt{2a}.$$

Two radicals, which do not appear to be similar, may become so by simplification (Art. 104).

For example,

$$\sqrt{48ab^2} + b\sqrt{75a} = 4b\sqrt{3a} + 5b\sqrt{3a} = 9b\sqrt{3a};$$

Also, $2\sqrt{45} - 3\sqrt{5} = 6\sqrt{5} - 3\sqrt{5} = 3\sqrt{5}$.

When the radicals are not similar, their addition or subtraction can only be indicated.

Thus, to add $3\sqrt{b}$ to $5\sqrt{a}$, we write,

$$5\sqrt{a} + 3\sqrt{b}.$$

Multiplication of Radical Quantities of the Second Degree.

107. Let $a\sqrt{b}$ and $c\sqrt{d}$ denote any two radicals of the second degree; their product will be denoted thus,

$$a\sqrt{b} \times c\sqrt{d},$$

which, since the order of the factors may be changed without altering the value of the product, may be written,

$$a \times c \times \sqrt{b} \times \sqrt{d}.$$

The product of the last factors from the 1st principle of Art. 104, is equal to $\sqrt{bd}$; we have, therefore,

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}.$$

Hence, to multiply one radical of the second degree by another, we have the following

RULE.

Multiply the co-efficients together for a new co-efficient; after this write the radical sign, and under it the product of the quantities under both radical signs.

EXAMPLES.

1. $3\sqrt{5ab} \times 4\sqrt{20a} = 12\sqrt{100a^2b} = 120a\sqrt{b}.$

2. $2a\sqrt{bc} \times 3a\sqrt{bc} = 6a^2\sqrt{b^2c^2} = 6a^2bc.$

3. $2a\sqrt{a^2 + b^2} \times -3a\sqrt{a^2 + b^2} = -6a^2(a^2 + b^2).$

Division of Radical Quantities of the Second Degree.

108. Let $a\sqrt{b}$ and $c\sqrt{d}$ represent any two radicals of the second degree, and let it be required to find the quotient of the first by the second. This quotient may be indicated thus,

$$\frac{a\sqrt{b}}{c\sqrt{d}}, \text{ which is equal to } \frac{a}{c} \times \frac{\sqrt{b}}{\sqrt{d}};$$

but from the 2d principle of Art. 104,

$$\frac{\sqrt{b}}{\sqrt{d}} = \sqrt{\frac{b}{d}}; \quad \text{hence} \quad \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}.$$

Hence, to divide one radical of the second degree by another, we have the following

RULE.

Divide the co-efficient of the dividend by the co-efficient of the divisor for a new co-efficient; after this, write the radical sign, placing under it the quotient obtained by dividing the quantity under the radical sign in the dividend by that in the divisor.

For example, $5a\sqrt{b} \div 2b\sqrt{c} = \frac{5a}{2b}\sqrt{\frac{b}{c}};$

And, $12ac\sqrt{6bc} \div 4c\sqrt{2b} = 3a\sqrt{\frac{6bc}{2b}} = 3a\sqrt{3c}.$

109. The following transformation is of frequent application in finding an approximate value for a radical expression of a particular form.

Having given an expression of the form,

$$\frac{a}{p + \sqrt{q}} \quad \text{or} \quad \frac{a}{p - \sqrt{q}},$$

in which a and p are any numbers whatever, and q not a perfect square, it is the object of the transformation to render the denominator a rational quantity.

This object is attained by multiplying both terms of the fraction by $p-\sqrt{q}$, when the denominator is $p+\sqrt{q}$, and by $p+\sqrt{q}$, when the denominator is $p-\sqrt{q}$; and recollecting that the sum of two quantities, multiplied by their difference, is equal to the difference of their squares: hence,

$$\frac{a}{p+\sqrt{q}}=\frac{a(p-\sqrt{q})}{(p+\sqrt{q})(p-\sqrt{q})}=\frac{a(p-\sqrt{q})}{p^2-q}=\frac{ap-a\sqrt{q}}{p^2-q}:$$

$$\frac{a}{p-\sqrt{q}}=\frac{a(p+\sqrt{q})}{(p-\sqrt{q})(p+\sqrt{q})}=\frac{a(p+\sqrt{q})}{p^2-q}=\frac{ap+a\sqrt{q}}{p^2-q}$$

in which the denominators are rational.

As an example to illustrate the utility of this method of approximation, let it be required to find the approximate value of the expression $\frac{7}{3-\sqrt{5}}$. We write

$$\frac{7}{3-\sqrt{5}}=\frac{7(3+\sqrt{5})}{9-5}=\frac{21+7\sqrt{5}}{4}.$$

But, $\quad 7\sqrt{5}=\sqrt{49\times 5}=\sqrt{245}=15$ to within less than 1. Therefore,

$$\frac{7}{3-\sqrt{5}}=\frac{21+15 \text{ to within less than } 1}{4}=9 \text{ to within}$$

less than $\frac{1}{4}$; hence, 9 differs from the true value by less than *one fourth.*

If we wish a more exact value for this expression, *extract the square root of* 245 *to a certain number of decimal places, add* 21 *to this root, and divide the result by* 4.

Take the expression, $\quad \dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}},$

and find its value to within less than 0.01.

We have,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}} = \frac{7\sqrt{5}(\sqrt{11}-\sqrt{3})}{11-3} = \frac{7\sqrt{55}-7\sqrt{15}}{8}.$$

Now, $7\sqrt{55} = \sqrt{55 \times 49} = \sqrt{2695} = 51.91$, within less than 0.01,

and $7\sqrt{15} = \sqrt{15 \times 49} = \sqrt{735} = 27.11$; . . . ;

therefore,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}} = \frac{51.91-27.11}{8} = \frac{24.80}{8} = 3.10.$$

Hence, we have 3.10 for the required result. This is true to within less than $\frac{1}{800}$.

By a similar process, it may be found, that,

$$\frac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{5}} = 2.123, \text{ is exact to within less than } 0.001.$$

REMARK.—The value of expressions similar to those above, may be calculated by approximating to the value of each of the radicals which enter the numerator and denominator. But as the value of the denominator would not be exact, we could not determine the degree of approximation which would be obtained, whereas by the method just indicated, the denominator becomes *rational*, and we always know to what degree of accuracy the approximation is made.

PROMISCUOUS EXAMPLES.

1. Simplify $\sqrt{125}$. *Ans.* $5\sqrt{5}$.

2. Reduce $\sqrt{\frac{50}{147}}$ to its simplest form.

We observe that 25 will divide the numerator, and hence,

$$\sqrt{\frac{50}{147}} = \sqrt{\frac{25 \times 2}{147}} = 5\sqrt{\frac{2}{147}}.$$

Since the perfect square 49 will divide 147,

$$5\sqrt{\frac{2}{147}} = 5\sqrt{\frac{2}{49 \times 3}} = \frac{5}{7}\sqrt{\frac{2}{3}}$$

Divide the coefficient of the radical by 3, and multiply the number under the radical by the square of 3; then,

$$\frac{5}{7}\sqrt{\frac{2}{3}} = \frac{5}{21}\sqrt{\frac{18}{3}} = \frac{5}{21}\sqrt{6}.$$

3. Reduce $\sqrt{98a^2x}$ to its most simple form.

Ans. $7a\sqrt{2x}$.

4. Reduce $\sqrt{(x^3 - a^2x^2)}$ to its most simple form.

5. Required the sum of $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

6. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$.

7. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.

Ans. $\frac{19}{30}\sqrt{6}$.

8. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.

9. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{4}{45}\sqrt{15}$

11. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

12. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

13. Divide $6\sqrt{10}$ by $3\sqrt{5}$.

14. What is the sum of $\sqrt{48ab^2}$ and $b\sqrt{75a}$.

15. What is the sum of $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.

Ans. $(3a^2b + 5ab)\sqrt{2ab}$.

CHAPTER VI.

EQUATIONS OF THE SECOND DEGREE.

110. Equations of the second degree may involve *but one* unknown quantity, or they may involve *more than one.*

We shall first consider the former class.

111. An equation containing but one unknown quantity is said to be of the *second degree,* when the highest power of the unknown quantity in any term, is the second.

Let us assume the equation,

$$\frac{a}{b}x^2 - cx + d = cx^2 + \frac{b}{d}x + a.$$

Clearing of fractions,

$$adx^2 - bcdx + bd^2 = bcdx^2 + b^2x + abd\,;$$

transposing, $\quad adx^2 - bcdx^2 - bcdx - b^2x = abd - bd^2\,;$

factoring, $\quad (ad - bcd)x^2 - (bcd + b^2)x = abd - bd^2\,;$

dividing both members by the co-efficient of x^2,

$$x^2 - \frac{bcd + b^2}{ad - bcd}x = \frac{abd - bd^2}{ad - bcd}.$$

If we now replace the co-efficient of x by $2p$, and the second member by q, we shall have

$$x^2 + 2px = q\,;$$

and since every equation of the second degree may be reduced, in like manner, we conclude that, every equation of the second degree, involving but one unknown quantity, can be reduced to the form

$$x^2 + 2px = q,$$

by the following

RULE.

I. *Clear the equation of fractions;*

II. *Transpose all the known terms to the second member, and all the unknown terms to the first.*

III. *Reduce the terms involving the square of the unknown quantity to a single term of two factors, one of which is the square of the unknown quantity;*

IV. *Then, divide both members by the co-efficient of the square of the unknown quantity.*

112. If $2p$, the algebraic sum of the co-efficients of the first powers of x, becomes equal to 0, the equation will take the form

$$x^2 = q,$$

and this is called, an *incomplete equation* of the second degree. Hence,

An incomplete equation of the second degree involves only the second power of the unknown quantity and known terms, and may be reduced to the form

$$x^2 = q.$$

Solution of Incomplete Equations.

113. Having reduced the equation to the required form, we have simply *to extract the square root of both members to find the value of the unknown quantity.*

Extracting the square root of both members of the equation

$$x^2 = q, \quad \text{we have} \quad x = \sqrt{q}.$$

If q is a perfect square, the exact value of x can be found by extracting the square root of q, and the value of x will then be expressed either algebraically or in numbers.

If q is an algebraic quantity, and not a perfect square, it must be reduced to its simplest form by the rules for reducing radicals of the second degree. If q is a number, and not a perfect square, its square root must be determined, approximately, by the rules already given.

But the *square* of any number is $+$, whether the number itself have the $+$ or $-$ sign: hence, it follows that

$$(+\sqrt{q})^2 = q, \quad \text{and} \quad (-\sqrt{q})^2 = q;$$

and therefore, the unknown quantity x is susceptible of two distinct values, viz:

$$x = +\sqrt{q}, \quad \text{and} \quad x = -\sqrt{q};$$

and either of these values, being substituted for x, will satisfy the given equation. For,

$$x^2 = +\sqrt{q} \times +\sqrt{q} = q;$$

and $\quad x^2 = -\sqrt{q} \times -\sqrt{q} = q;\quad$ hence,

Every incomplete equation of the second degree has two roots which are numerically equal to each other; one having the sign plus, and the other the sign minus (Art. 77).

EXAMPLES.

1. Let us take the equation

$$\frac{1}{3}x^2 - 3 + \frac{5}{12}x^2 = \frac{7}{24} - x^2 + \frac{299}{24}.$$

which, by making the terms entire, becomes

$$8x^2 - 72 + 10x^2 = 7 - 24x^2 + 299,$$

and by transposing and reducing

$$42x^2 = 378 \quad \text{and} \quad x^2 = \frac{378}{42} = 9;$$

hence, $\quad x = +\sqrt{9} = +3; \quad$ and $\quad x = -\sqrt{9} = -3.$

2. As a second example, let us take the equation

$$3x^2 = 5.$$

Dividing both members by 3 and extracting the square root,

$$x = \pm\sqrt{\frac{5}{3}} = \pm\frac{1}{3}\sqrt{15};$$

in which the values of x must be determined approximately

3. What are the values of x in the equation

$$11(x^2 - 4) = 5(x^2 + 2). \qquad \textit{Ans. } x = \pm 3.$$

4. What are the values of x in the equation

$$\frac{\sqrt{m^2 - x^2}}{x} = n. \qquad \textit{Ans } x = \pm\frac{m}{\sqrt{1 + n^2}}.$$

Solution of Equations of the Second Degree.

114. Let us now solve the equation of the second degree

$$x^2 + 2px = q.$$

If we compare the first member with the square of

$$x + p, \quad \text{which is} \quad x^2 + 2px + p^2,$$

we see, that it needs but the square of p to render it a perfect square. If then, p^2 be added to the first member, it will become a perfect square; but in order to preserve the equality of the members, p^2 must also be added to the second member. Making these additions, we have

$$x^2 + 2px + p^2 = q + p^2;$$

this is called, *completing the square*, and is done, *by adding the square of half the co-efficient of* x *to both members of the equation.*

Now, if we extract the square root of both members, we have,

$$x + p = \pm\sqrt{q + p^2},$$

and by transposing p, we shall have

$$x = -p + \sqrt{q + p^2}, \quad \text{and} \quad x = -p - \sqrt{q + p^2}.$$

Either of these values, being substituted for x in the equation

$$x^2 + 2px = q$$

will satisfy it. For, substituting the first value,

$$x^2 = (-p + \sqrt{q + p^2})^2 = p^2 - 2p\sqrt{q + p^2} + q + p^2.$$

and

$$2px = 2p \times (-p + \sqrt{q + p^2}) = -2p^2 + 2p\sqrt{q + p^2},$$

by adding $\qquad x^2 + 2px = q.$

Substituting the second value of x, we find,

$$x^2 = (-p - \sqrt{q + p^2})^2 = \quad p^2 + 2p\sqrt{q + p^2} + q + p^2,$$

and

$$2px = 2p(-p - \sqrt{q + p^2}) = -2p^2 - 2p\sqrt{q + p^2};$$

by adding $\qquad x^2 + 2px = q;$

and consequently, both values found above, are *roots* of the equation.

In order to refer readily, to either of these values, we shall call the one which arises from using the + sign before the radical, the *first value* of x, or the *first root* of the equation; and the other, the *second value* of x, or the *second root* of the equation.

Having reduced a complete equation of the second degree to the form

$$x^2 + 2px = q,$$

we can write immediately the two values of the unknown quantity by the following

RULE.

I. *The first value of the unknown quantity is equal to half the co-efficient of x, taken with a contrary sign, plus the square root of the second member increased by the square of half this co-efficient.*

II. *The second value is equal to half the co-efficient of x, taken with a contrary sign, minus the square root of the second member increased by the square of half this co-efficient.*

EXAMPLES.

1. Let us take as an example,

$$x^2 - 7x + 10 = 0.$$

Reducing to required form,

$$x^2 - 7x = -10;$$

whence by the rule, $x = \frac{7}{2} + \sqrt{-10 + \frac{49}{4}} = 5;$

and, $x = \frac{7}{2} - \sqrt{-10 + \frac{49}{4}} = 2.$

2. As a second example, let us take the equation

$$\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}.$$

Reducing to the required form, we have,

$$x^2+\frac{2}{22}x=\frac{360}{22};$$

whence, $$x=-\frac{1}{22}+\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}$$

and $$x=-\frac{1}{22}-\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}.$$

It often occurs, in the solution of equations, that p^2 and q are fractions, as in the above example. These fractions most generally arise from dividing by the co-efficient of x^2 in the reduction of the equation to the required form. When this is the case, we readily discover the quantity by which it is necessary to multiply the term q, in order to reduce it to the same denominator with p^2; after which, the numerators may be added together and placed over the common denominator. After this operation, the denominator will be a perfect square, and may be brought from under the radical sign, and will become a divisor of the square root of the numerator.

To apply these principles in reducing the radical part of the values of x, in the last example, we have

$$\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}=\pm\sqrt{\frac{360\times 22}{(22)^2}+\frac{1}{(22)^2}}=\pm\sqrt{\frac{7920+1}{(22)^2}}$$

$$=\pm\frac{1}{22}\sqrt{7921}=\pm\frac{89}{22};$$

and therefore, the two values of x become,

$$x=-\frac{1}{22}+\frac{89}{22}=\frac{88}{22}=4;$$

and $$x=-\frac{1}{22}-\frac{89}{2}=-\frac{90}{22}=-\frac{45}{11};$$

either of which being substituted for x in the given equation, will satisfy it.

3. What are the values of x in the equation

$$ax^2-ac=cx-bx^2$$

Reducing to required form, we have,

$$x^2 - \frac{c}{a+b}x = \frac{ac}{a+b};$$

whence, $$x = +\frac{c}{2(a+b)} + \sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}},$$

and, $$x = +\frac{c}{2(a+b)} - \sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}}.$$

Reducing the terms under the radical sign to a common denominator, we find,

$$\sqrt{\frac{ac}{a+b} + \frac{c^2}{4(a+b)^2}} = \sqrt{\frac{4a^2c + 4abc + c^2}{4(a+b)^2}} = \frac{\sqrt{4a^2c + 4abc + c^2}}{2(a+b)};$$

hence, $$x = \frac{c \pm \sqrt{4a^2c + 4abc + c^2}}{2(a+b)}.$$

4. What are the values of x, in the equation,

$$6x^2 - 37x = -57.$$

By reducing to the required form, we have,

$$x^2 - \frac{37}{6}x = -\frac{57}{6};$$

whence, $$x = +\frac{37}{12} \pm \sqrt{-\frac{57}{6} + \left(\frac{37}{12}\right)^2}$$

Reducing the quantities under the radical sign to a common denominator, we have,

$$x = +\frac{37}{12} \pm \sqrt{\frac{-114 \times 12}{(12)^2} + \frac{(37)^2}{(12)^2}}.$$

But, $114 \times 12 = 1368$; and $(37)^2 = 1369$;

hence, $$x = +\frac{37}{12} \pm \sqrt{\frac{-1368 + 1369}{(12)^2}} = +\frac{37}{12} \pm \frac{1}{12};$$

or, $$x = +\frac{37}{12} + \frac{1}{12} = \frac{19}{6},$$

and, $$x = +\frac{37}{12} - \frac{1}{12} = 3.$$

5. What are the values of x, in the equation,

$$4a^2 - 2x^2 + 2ax = 18ab - 18b^2.$$

Reducing to the required form, we have,

$$x^2 - ax = 2a^2 - 9ab + 9b^2;$$

whence,
$$x = \frac{a}{2} \pm \sqrt{2a^2 - 9ab + 9b^2 + \frac{a^2}{4}}$$
$$= \frac{a}{2} \pm \sqrt{\frac{9a^2}{4} - 9ab + 9b^2}.$$

The radical part is equal to $\frac{3a}{2} - 3b$; hence,

$$x = \frac{a}{2} \pm \left(\frac{3a}{2} - 3b\right); \quad \text{or} \quad \begin{cases} x = 2a - 3b. \\ x = -a + 3b. \end{cases}$$

Find the values of x in the following

EXAMPLES.

1. $\frac{x^2}{3} - \frac{a}{b}x = 1 - \frac{b}{a}x - \frac{2x^2}{3}$. *Ans.* $x = \frac{a}{b}$, $x = -\frac{b}{a}$.

2. $\frac{dx}{c} + \frac{3x^2}{4} + 1 = \frac{1+c}{c} - \frac{x^2}{4} + \frac{x}{d}$.

Ans. $x = \frac{1}{d}$, $x = -\frac{d}{c}$

3. $\frac{x^2}{4} - \frac{2x}{3} + \frac{59}{8} = 8 - \frac{x^2}{4} - \frac{x}{3}$.

Ans. $x = \frac{3}{2}$, $x = -\frac{5}{6}$

4. $\frac{90}{x} - \frac{90}{x+1} = \frac{27}{x+2}$. *Ans.* $x = 4$, $x = -\frac{5}{3}$

5. $\frac{2x-10}{8-x} - 2 = \frac{x+3}{x-2}$. *Ans.* $x = 7$, $x = \frac{4}{5}$

6. $ax - \frac{x^2}{b} + b = \frac{b-1}{b}x^2 + \frac{b}{a}x$. *Ans.* $x = a$, $x = -\frac{b}{a}$

7. $\frac{a-b}{c}x + \frac{3x^2}{2} - \frac{a^2}{c^2} = \frac{b+a}{c}x + \frac{x^2}{2} - \frac{b^2}{c^2}$.

Ans. $x = \frac{b+a}{c}$, $x = \frac{b-a}{c}$.

8. $mx^2 + mn = 2m\sqrt{n}\,x + nx^2$.

Ans. $x = \dfrac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}}, \quad x = \dfrac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$.

9. $abx^2 - \dfrac{6a^2}{c^2} + \dfrac{b^2x}{c} = \dfrac{ab - 2b^2}{c^2} - \dfrac{3a^2}{c}x$.

Ans. $x = \dfrac{2a - b}{ac}, \quad x = -\dfrac{3a + 2b}{bc}$.

10. $\dfrac{4x^2}{7} + \dfrac{2x}{7} + 10 = 19 - \dfrac{3x^2}{7} + \dfrac{58x}{7}$.

Ans. $x = 9, \quad x = -1$.

11. $\dfrac{x + a}{x - a} - b = \dfrac{a - x}{a + x}$. *Ans.* $x = \pm a\sqrt{\dfrac{b + 2}{b - 2}}$.

12. $2x + 2 = 24 - 5x - 2x^2$. *Ans.* $x = 2, \quad x = -\dfrac{11}{2}$.

13. $x^2 - x - 40 = 170$. *Ans.* $x = 15$, and $x = -14$.

14. $3x^2 + 2x - 9 = 76$. *Ans.* $x = 5$, and $x = -5\frac{2}{3}$.

15. $a^2 + b^2 - 2bx + x^2 = \dfrac{m^2x^2}{n^2}$.

Ans. $x = \dfrac{n}{n^2 - m^2}(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2})$.

Problems giving rise to Equations of the Second Degree involving but one unknown quantity.

1. Find a number such that three times the number added to twice its square will be equal to 65.

Let x denote the number. Then from the conditions,

$$2x^2 + 3x = 65 \quad \cdots \quad (1)$$

Whence,
$$x = -\frac{3}{4} \pm \sqrt{\frac{65}{2} + \frac{9}{16}};$$

reducing
$$x = 5 \quad \text{and} \quad x = -\frac{13}{2}.$$

Both of these roots verify the equation: for,

$$2 \times (5)^2 + 3 \times 5 = 2 \times 25 + 15 = 65;$$

and $$2\left(-\frac{13}{2}\right)^2 + 3 \times -\frac{13}{2} = \frac{169}{2} - \frac{39}{2} = \frac{130}{2} = 65.$$

The first root satisfies the conditions of the problem as enunciated.

The second root will also satisfy the conditions, if we regard its algebraic sign. Had we denoted the unknown quantity by $-x$, we should have found

$$2x^2 - 3x = 65 \quad - \; - \; - \quad (2)$$

from which $$x = \frac{13}{2} \quad \text{and} \quad x = -5.$$

We see that the roots of this equation differ from those of equation (1) only in their signs, a result which was to have been expected, since we can change equation (1) into equation (2) by simply changing the sign of x, and the reverse.

2. A person purchased a number of yards of cloth for 240 cents. If he had received three yards less, for the same sum, it would have cost him 4 cents more per yard. How many yards did he purchase?

Let x denote the number of yards purchased.

Then will $\frac{240}{x}$ denote the number of cents paid per yard.

Had he received three yards less,

$x - 3$, would have denoted the number of yards purchased, and

$\frac{240}{x-3}$, would have denoted the number of cents he paid per yard.

From the conditions of the problem,

$$\frac{240}{x-3} - \frac{240}{x} = 4;$$

by reducing, $$x^2 - 3x = 180$$

whence, $$x = 15 \quad \text{and} \quad x = -12.$$

The value $x = 15$ satisfies the conditions of the problem, understood in their arithmetical sense; for, 15 yards for 240

cents, gives $\frac{240}{15}$, or 16 cents for the price of one yard, and 12 yards for 240 cents, gives 20 cents for the price of one yard, which exceeds 16 by 4.

The value $+x = -12$, or $-x = +12$, will satisfy the conditions of the following problem:

A person sold a number of yards of cloth for 240 cents: if he had received the same sum for 3 yards more, it would have brought him 4 cents less per yard. How many yards did he sell?

If we denote the number of yards sold by x, the statement of this last problem, and the given one, both give rise to the same equation,

$$x^2 - 3x = 180,$$

hence, the solution of this equation ought to give the answers to both problems, as we see that it does.

Generally, when the solution of the equation of a problem gives two roots, if the problem does not admit of two solutions there is always another problem whose statement gives rise to the same equation as the given one, and in this case the two roots form answers to both problems.

3. A man bought a horse, which he sold for 24 dollars. At the sale, he lost as much per cent. on the price of his purchase, as the horse cost him. What did he pay for the horse?

Let x denote the number of dollars that he paid for the horse: then, $x - 24$ will denote the number of dollars that he lost.

But as he lost x per cent. by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he lost a number of dollars denoted by $\frac{x^2}{100}$; we have then the equation

$$\frac{x^2}{100} = x - 24, \quad \text{whence} \quad x^2 - 100x = -2400;$$

Therefore, $\quad x = 60 \quad$ and $\quad x = 40.$

Both of these values satisfy the conditions of the problem.

For, in the first place, suppose the man gave 60 dollars for the horse and sold him for 24, he then loses 36 dollars. But, from the enunciation, he should lose 60 *per cent.* of 60, that is,

$$\frac{60}{100} \text{ of } 60 = \frac{60 \times 60}{100} = 36;$$

therefore, 60 satisfies the problem.

If he pays 40 dollars for the horse, he loses 16 by the sale; for, he should lose 40 *per cent.* of 40, or

$$40 \times \frac{40}{100} = 16;$$

therefore, 40 satisfies the conditions of the problem.

4. A grazier bought as many sheep as cost him £60, and after reserving 15 out of the number, he sold the remainder for £54, and gained 2*s.* a head on those he sold: how many did he buy? *Ans.* 75.

5. A merchant bought cloth for which he paid £33 15*s.*, which he sold again at £2 8*s.* per piece, and gained by the bargain as much as one piece cost him: how many pieces did he buy? *Ans.* 15.

6. What number is that, which, being divided by the product of its digits, the quotient will be 3; and if 18 be added to it, the order of its digits will be reversed? *Ans.* 24.

7. Find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product will be 21. *Ans.* 7 or 3.

8. Two persons, A and B, departed from different places at the same time, and traveled towards each other. On meeting, it appeared that A had traveled 18 miles more than B; and that A could have performed B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. How far did each travel?

Ans. $\begin{cases} \text{A 72 miles.} \\ \text{B 54 miles.} \end{cases}$

9. A company at a tavern had £8 15*s.* to pay for their reckoning; but before the bill was settled, two of them left

the room, and then those who remained had 10*s.* apiece more to pay than before: how many were there in the company?

Ans. 7.

10. What two numbers are those whose difference is 15, and of which the cube of the lesser is equal to half their product?

Ans. 3 and 18.

11. Two partners, A and B, gained \$140 in trade: A's money was 3 months in trade, and his gain was \$60 less than his stock: B's money was \$50 more than A's, and was in trade 5 months: what was A's stock? *Ans.* \$100.

12. Two persons, A and B, start from two different points, and travel toward each other. When they meet, it appears that A has traveled 30 miles more than B. It also appears that it will take A 4 days to travel the road that B had come, and B 9 days to travel the road that A had come. What was their distance apart when they set out? *Ans.* 150 miles.

Discussion of Equations of the Second Degree involving but one unknown quantity.

115. It has been shown that every complete equation of the second degree can be reduced to the form (Art. 113)

$$x^2 + 2px = q \quad - \; - \; - \quad (1),$$

in which p and q are numerical or algebraic, entire or fractional, and their signs plus or minus.

If we make the first member a perfect square, by completing the square (Art. 112*), we have

$$x^2 + 2px + p^2 = q + p^2,$$

which may be put under the form

$$(x + p)^2 = q + p^2.$$

Now, whatever may be the value of $q + p^2$, its square root may be represented by m, and the equation put under the form

$$(x + p)^2 = m^2, \quad \text{and consequently,} \quad (x + p)^2 - m^2 = 0.$$

But, as the first member of the last equation is the difference between two squares, it may be put under the form

$$(x + p - m)(x + p + m) = 0 \quad . \quad . \quad . \quad (2),$$

in which the first member is the product of two factors, and the second 0. Now, we can make this product equal to 0, and consequently satisfy equation (2) only in two different ways, viz., by making

$$x + p - m = 0, \quad \text{whence,} \quad x = -p + m,$$

or, by making

$$x + p + m = 0, \quad \text{whence,} \quad x = -p - m.$$

Now, either of these values being substituted for x in equation (2), will satisfy that equation, and consequently, will satisfy equation (1), from which it was derived. Hence, we conclude,

1st. *That every equation of the second degree has two roots, and only two.*

2d. *That the first member of every equation of the second degree, whose second member is 0, can be resolved into two binomial factors of the first degree with respect to the unknown quantity, having the unknown quantity for a first term and the two roots, with their signs changed, for second terms.*

For example, the equation

$$x^2 + 3x - 28 = 0$$

being solved, gives

$$x = 4 \quad \text{and} \quad x = -7;$$

either of which values will satisfy the equation. We also have

$$(x - 4)(x + 7) = x^2 + 3x - 28 = 0.$$

If the roots of an equation are known, we can readily form the binomial factors and deduce the equation.

EXAMPLES.

1. What are the factors, and what is the equation, of which the roots are 8 and -9?

Ans. $x - 8$ and $x + 9$ are the binomial factors,

and $x^2 + x - 72 = 0$ is the equation.

2. What are the factors, and what is the equation, of which the roots are -1 and $+1$?

$x+1$ and $x-1$ are the factors,

and $x^2-1=0$ is the equation.

3. What are the factors, and what is the equation, whose roots are

$$\frac{7+\sqrt{-1039}}{16} \text{ and } \frac{7-\sqrt{-1039}}{16}?$$

Ans. $\left(x-\frac{7+\sqrt{-1039}}{16}\right)$ and $\left(x-\frac{7-\sqrt{-1039}}{16}\right)$

are the factors,

and $8x^2-7x+34=0$ is the equation.

116. If we designate the two roots, found in the preceding article, by x' and x'', we shall have,

$$x' = -p+m,$$

$$x'' = -p-m;$$

or substituting for m its value $\sqrt{q+p^2}$,

$$x' = -p+\sqrt{q+p^2},$$

$$x'' = -p-\sqrt{q+p^2}.$$

Adding these equations, member to member, we get

$$x'+x'' = -2p;$$

and multiplying them, member by member, and reducing, we find

$$x'x'' = -q.$$

Hence, after an equation has been reduced to the form of

$$x^2+2px=q,$$

1st. *The algebraic sum of its two roots is equal to the co-efficient of the first power of the unknown quantity, with its sign changed.*

2d. *The product of the two roots is equal to the second member with its sign changed.*

If the sum of two quantities is given or known, their product will be the greatest possible when they are equal.

Let $2p$ be the sum of two quantities, and denote their difference by $2d$; then,

$p + d$ will denote the greater, and $p - d$ the less quantity.

If we represent their product by q, we shall have

$$p^2 - d^2 = q.$$

Now, it is plain that q will increase as d diminishes, and that it will be the greatest possible, when $d = 0$; that is, when the two quantities are equal to each other, in which case the product becomes equal to p^2. Hence,

3d. *The greatest possible value of the product of the two roots, is equal to the square of half the co-efficient of the first power of the unknown quantity.*

Of the Four Forms.

117. Thus far, we have regarded p and q as algebraic quantities, without considering the essential sign of either, nor have we at all regarded their relative values.

If we first suppose p and q to be both essentially positive, then to become negative in succession, and after that, both to become negative together, we shall have all the combinations of signs which can arise. The complete equation of the second degree will, therefore, always be expressed under one of the four following forms:—

$$x^2 + 2px = \quad q \qquad (1),$$
$$x^2 - 2px = \quad q \qquad (2),$$
$$x^2 + 2px = -q \qquad (3),$$
$$x^2 - 2px = -q \qquad (4).$$

These equations being solved, give

$$x = -p \pm \sqrt{\quad q + p^2} \qquad (1),$$
$$x = +p \pm \sqrt{\quad q + p^2} \qquad (2),$$
$$x = -p \pm \sqrt{-q + p^2} \qquad (3),$$
$$x = +p \pm \sqrt{-q + p^2} \qquad (4).$$

In the first and second forms, the quantity under the radical sign will be positive, whatever be the relative values of p and q, since q and p^2 are both positive; and therefore, both roots will be real. And since

$$q + p^2 > p^2, \quad \text{it follows that,} \quad \sqrt{q + p^2} > p,$$

and consequently, *the roots in both these forms will have the same signs as the radicals.*

In the *first* form, the first root will be positive and the second negative, the negative root being numerically the greater.

In the *second* form, the first root is positive and the second negative, the positive root being numerically the greater

In the third and fourth forms, if

$$p^2 > q,$$

the roots will be real, and since

$$p > \sqrt{-q + p^2},$$

they will have the same sign as the entire part of the root. Hence, *both roots will be negative in the third form, and both positive in the fourth.*

If $p^2 = q$, the quantity under the radical sign becomes 0, and the two values of x in both the third and fourth forms will be equal to each other; both equal to $-p$ in the third form, and both equal to $+p$ in the fourth.

If $p^2 < q$, the quantity under the radical sign is negative, and all the roots in the third and fourth forms are imaginary.

But from the third principle demonstrated in Art. 116, the greatest value of the product of the two roots is p^2, and from the second principle in the same article, this product is equal to q; hence, the supposition of $p^2 < q$ is absurd, and the values of the roots corresponding to the supposition ought to be *impossible* or *imaginary.*

When any particular supposition gives rise to imaginary results, we interpret these results as indicating that the supposition is absurd or impossible.

If $p = 0$, the roots in each form become equal with contrary signs; real in the first and second forms, and imaginary in the third and fourth.

If $q = 0$, the first and third forms become the same, as also, the second and fourth.

In the former case, the first root is equal to 0, and the second root is equal to $-2p$; in the latter case, the first root is equal to $+2p$, and the second to 0.

If $p = 0$ and $q = 0$, all the roots in the four forms reduce to 0.

In the preceding discussion we have made

$$p^2 > q, \quad p^2 < q, \quad \text{and} \quad p^2 = q;$$

we have also made p and q separately equal to 0, and then both equal to 0 at the same time.

These suppositions embrace every possible hypothesis that can be made upon p and q.

118. The results deduced in article 117 might have been obtained by a discussion of the *four forms* themselves, instead of their roots, making use of the principles demonstrated in article 116.

In the *first* form the product of the two roots is equal to $-q$, hence the roots must have contrary signs; their sum is $-2p$, hence the negative root is numerically the greater.

In the *second* form the product of the roots is equal to $-q$ and their sum equal to $+2p$; hence, their signs are unlike, and the positive root is the greater.

In the *third* form the product of the roots is equal to $+q$; hence, their signs are alike, and their sum being equal to $-2p$, they are both negative.

In the *fourth* form the product of the roots is equal to $+q$, and their sum is equal to $+2p$; hence, their signs are alike and both positive.

If $p = 0$, the sum of the roots must be equal to 0; or the roots must be equal with contrary signs.

If $q = 0$, the product of the roots is equal to 0; hence, one of the roots must be 0, and the other will be equal to the coefficient of the first power of the unknown quantity, taken with a contrary sign.

If $p = 0$ and $q = 0$, the sum of the roots must be equal to 0, and their product must be equal to 0; hence, the roots themselves must both be 0.

119. There is a singular case, sometimes met with in the discussion of problems, giving rise to equations of the second degree, which needs explanation.

To discuss it, take the equation

$$ax^2 + bx = c,$$

which gives
$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

If, now, we suppose $a = 0$, the expression for the value of x becomes

$$x = \frac{-b \pm b}{0}, \quad \text{whence,} \quad \begin{cases} x = \dfrac{0}{0}, \\ x = -\dfrac{2b}{0} = \infty. \end{cases}$$

But the supposition $a = 0$, reduces the given equation to $bx = c$, which is an equation of the *first* degree.

The roots, found above, however, admit of interpretation.

The first one reduces to the form $\dfrac{0}{0}$ in consequence of the existence of a factor, in both numerator and denominator, which factor becomes 0 for the particular supposition. To deduce the true value of the root, in this case, take

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a},$$

and multiply both terms of the fraction by $-b - \sqrt{b^2 + 4ac}$; after striking out the common factor $-2a$ we shall have

$$x = \frac{2c}{b + \sqrt{b^2 + 4ac}},$$

in which, if we make $a = 0$, the value of x reduces to $\frac{c}{b}$, the same value that we should obtain by solving the simple equation $bx = c$.

The other root ∞, is the value towards which the expression, for the second value of x, continually approaches as a is made smaller and smaller. It indicates that the equation, under the supposition, admits of but one root in finite terms. This should be the case, since the equation then becomes of the first degree.

120. The discussion of the following problem presents most of the circumstances usually met with in problems giving rise to equations of the second degree. In the solution of this problem, we employ the following principle of optics, viz. :—

The intensity of a light at any given distance, is equal to its intensity at the distance 1, *divided by the square of that distance.*

Problem of the Lights.

121. Find upon the line which joins two lights, A and B, of different intensities, the point which is equally illuminated by the lights.

Let A be assumed as the origin of distances, and regard all distances measured from A to the right as positive.

Let c represent the distance AB, between the two lights ; a the intensity of the light A at the distance 1, and b, the intensity of the light B at the distance 1.

Denote the distance AC, from A to the point of equal illumination, by x ; then will the distance from B to the same point be denoted by $c - x$.

From the principle assumed in the last article, the intensity of the light A, at the distance 1, being a, its intensity at the distances 2, 3, 4, &c., will be $\frac{a}{4}$, $\frac{a}{9}$, $\frac{a}{16}$, &c. ; hence, at the distance x it will be expressed by $\frac{a}{x^2}$.

In like manner, the intensity of B at the distance $c - x$, is $\frac{b}{(c-x)^2}$; but, by the conditions of the problem, these two intensities are equal to each other, and therefore we have the equation

$$\frac{a}{x^2} = \frac{b}{(c-x)^2};$$

which can be put under the form

$$\frac{(c-x)^2}{x^2} = \frac{b}{a};$$

hence, $$\frac{c-x}{x} = \frac{\pm\sqrt{b}}{\sqrt{a}}; \quad \text{whence}$$

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} \quad \text{- - - -} \quad (1),$$

$$x = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}} \quad \text{- - - -} \quad (2).$$

Since both of these values of x are always real, we conclude that there will be two points of equal illumination on the line AB, or on the line produced. Indeed, it is plain that there should be, not only a point of equal illumination between the lights, but also one on the prolongation of the line joining the lights and on the side of the lesser one.

To discuss these two values of x.

First, suppose $a > b$.

The first value of x is positive; and since

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < 1,$$

this value will be less than c, and consequently, the first point C, will be situated between the points A and B. We see, moreover, that the point will be nearer B than A; for, since $a > b$, we have

$$\sqrt{a}+\sqrt{a} \quad \text{or,} \quad 2\sqrt{a} > (\sqrt{a}+\sqrt{b}), \quad \text{whence}$$

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2}; \quad \text{and consequently,} \quad \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{c}{2}.$$

The second value of x is also positive; but since

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > 1,$$

it will be greater than c; and consequently, the second point will be at some point C', on the prolongation of AB, and at the right of the two lights.

This is as it should be; for, since the light at A is most intense, the point of equal illumination, between the lights, ought to be nearest the light B; and also, the point on the prolongation of AB ought to be on the side of the lesser light B.

Second, suppose $a < b$.

The first value of x is positive; and since

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < 1$$

this value of x will be less than c; consequently, the first point will fall at some point C, to the right of A, and between A and B.

We see, moreover, that it will be nearer A than B; for, since $a < b$, we have

$$\sqrt{a}+\sqrt{b} > 2\sqrt{a}, \quad \text{and consequently,} \quad \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \frac{c}{2}.$$

The second value of x is essentially negative, since the numerator is positive, and the denominator essentially negative.

We have agreed to consider distances from A to the right positive; hence, in accordance with the rule already established for interpreting negative results, the second point of equal illumination will be found at C'', somewhere to the left of A.

This is as it should be, since, under the supposition, the light at B is most intense; hence, the point of equal illumination, between the two lights, should be nearest A, and the point in the prolongation of AB, should be on the side nearest the feebler light A.

Third, suppose $a = b$, and $c > 0$.

The first value of x is then positive, and equal to $\frac{c}{2}$ hence, the first point is midway between the two lights.

The second value of x becomes $\frac{c\sqrt{a}}{0} = \infty$, a result which indicates that there is no other point of illumination at a finite distance from A.

This interpretation is evidently correct; for, under the supposition made, the lights are equally intense, and consequently, the point midway between them ought to be equally illuminated. It is also plain, that there can be no other point on the line which will enjoy that property.

Fourth, suppose $b = a$ and $c = 0$.

The first value of x becomes, $\frac{0}{2\sqrt{a}} = 0$, hence the first point is at A.

The second value of x becomes, $\frac{0}{0}$, a result which indicates that there are an infinite number of other points which are equally illuminated.

These conclusions are confirmed by a consideration of the conditions of the problem. Under this supposition, the lights are equal in intensity, and coincide with each other at the point A. That point ought then to be equally illuminated by the lights, as ought, also, every other point of the line on which the lights are placed.

Fifth, suppose $a > b$, or $a < b$, and $c = 0$.

Under these suppositions, both values of x reduce to 0, which shows that both points of equal illumination coincide with the point A.

This is evidently the case, for, since a is not equal to b, and the lights coincide at A, it is plain that no other point than A can be equally illuminated by them.

The preceding discussion presents a striking example of the precision with which the algebraic analysis responds to all the relations which exist between the quantities that enter a problem.

EXAMPLES INVOLVING RADICALS OF THE SECOND DEGREE.

1. Given, $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$, to find the values of x.

By reducing to entire terms, we have,

$$x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2,$$

by transposing, $\qquad x\sqrt{a^2 + x^2} = a^2 - x^2,$

and by squaring both members, $\qquad a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4,$

whence, $\qquad 3a^2x^2 = a^4,$

and, $\qquad x = \pm\sqrt{\dfrac{a^2}{3}}.$

2. Given, $\sqrt{\dfrac{a^2}{x^2} + b^2} - \sqrt{\dfrac{a^2}{x^2} - b^2} = b$, to find the values of x.

By transposing, $\qquad \sqrt{\dfrac{a^2}{x^2} + b^2} = \sqrt{\dfrac{a^2}{x^2} - b^2} + b;$

squaring both members, $\qquad \dfrac{a^2}{x^2} + b^2 = \dfrac{a^2}{x^2} - b^2 + 2b\sqrt{\dfrac{a^2}{x^2} - b^2} + b^2;$

whence, $\qquad b^2 = 2b\sqrt{\dfrac{a^2}{x^2} - b^2}$, and $b = 2\sqrt{\dfrac{a^2}{x^2} - b^2};$

squaring both members, $\qquad b^2 = \dfrac{4a^2}{x^2} - 4b^2;$

and hence, $\qquad x^2 = \dfrac{4a^2}{5b^2}$, and $x = \pm\dfrac{2a}{b\sqrt{5}}.$

3. Given, $\dfrac{a}{x} + \sqrt{\dfrac{a^2 - x^2}{x^2}} = \dfrac{x}{b}$, to find the values of x.

Ans. $x = \pm\sqrt{2ab - b^2}.$

4. Given, $\sqrt{\dfrac{x + a}{a}} + 2\sqrt{\dfrac{a}{x + a}} = b^2\sqrt{\dfrac{x}{x + a}}$, to find the values of x.

Ans. $x = \dfrac{a}{(b \mp 1)^2}.$

5. Given, $\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$, to find the values of x.

$$Ans.\quad x = \pm\frac{2a\sqrt{b}}{1 + b}.$$

6. Given, $\dfrac{\sqrt{x} + \sqrt{x - a}}{\sqrt{x} - \sqrt{x - a}} = \dfrac{n^2 a}{x - a}$, to find the values of x.

$$Ans.\quad x = \frac{a(1 \pm n)^2}{1 \pm 2n}.$$

7. Given, $\dfrac{\sqrt{a + x}}{\sqrt{x}} + \dfrac{\sqrt{a - x}}{\sqrt{x}} = \sqrt{\dfrac{x}{b}}$, to find the values of x.

$$Ans.\quad x = \pm 2\sqrt{ab - b^2}.$$

8. Given, $\dfrac{a + x + \sqrt{2ax + x^2}}{a + x} = b$, to find the values of x.

$$Ans.\quad x = \frac{\pm a(1 \pm \sqrt{2b - b^2})}{\sqrt{2b - b^2}}.$$

Of Trinomial Equations.

122. A *trinomial equation* is one which involves only terms containing two different powers of the unknown quantity and a known term or terms.

123. Every trinomial equation can be reduced to the form

$$x^m + 2px^n = q \quad \text{- - - - - -} \quad (1),$$

in which m and n are positive whole numbers, and p and q known quantities, by means of a rule entirely similar to that given in article 111.

If we suppose $m = 2$ and $n = 1$, equation (1) becomes

$$x^2 + 2px = q,$$

a trinomial equation of the second degree.

124. The solution of trinomial equations of the second degree, has already been explained. The methods, there explained, are, with some slight modifications, applicable to all trinomial equations in which $m = 2n$, that is, to all equations of the form

$$x^{2n} + 2px^n = q.$$

To demonstrate a rule for the solution of equations of this form, let us place

$$x^n = y; \quad \text{whence,} \quad x^{2n} = y^2.$$

These values of x^n and x^{2n}, being substituted in the given equation, reduce it to

$$y^2 + 2py = q,$$

whence, $$y = -p \pm \sqrt{q + p^2},$$

or, $$x^n = -p \pm \sqrt{q + p^2}.$$

Now, the n^{th} root, of the first member, is x (Art. 18), and although we have not yet explained how to extract the n^{th} root of an algebraic quantity, we may indicate the n^{th} root of the second member. Hence, (axiom 6),

$$x = \sqrt[n]{-p \pm \sqrt{q + p^2}}.$$

Hence, to solve a trinomial equation which can be reduced to the form $x^{2n} + 2px^n = q$, we have the following

RULE.

Reduce the equation to the form of $x^{2n} + 2px^n = q$ *; the values of the unknown quantity will then be found by extracting the* n^{th} *root of half the co-efficient of the lowest power of the unknown quantity with its sign changed, plus or minus the square root of the second member increased by the square of half the co-efficient of the lowest power of the unknown quantity.*

If $n = 2$, the roots of the equation are of the form

$$x = \pm \sqrt{-p \pm \sqrt{q + p^2}}.$$

We see that the unknown quantity has four values, since each of the signs $+$ and $-$, which affect the first radical can be combined, in succession, with each of the signs which affect the second; *but these values, taken two and two, are numerically equal, and have contrary signs.*

EXAMPLES.

1. Take the equation

$$x^4 - 25x^2 = -144.$$

This being of the required form, we have by application of the rule,

$$x = \pm\sqrt{\frac{25}{2} \pm \sqrt{-144 + \tfrac{625}{4}}},$$

whence, $\quad x = \pm\sqrt{\dfrac{25}{2} \pm \dfrac{7}{2}}$;

hence, the four roots are $+4$, -4, $+3$, and -3.

2. As a second example, take the equation

$$x^4 - 7x^2 = 8.$$

Whence, by the rule,

$$x = \pm\sqrt{\frac{7}{2} \pm \sqrt{8 + \tfrac{49}{4}}} = \pm\sqrt{\frac{7}{2} \pm \frac{9}{2}};$$

hence, the four roots are,

$$+2\sqrt{2}, \quad -2\sqrt{2}, \quad +\sqrt{-1} \quad \text{and} \quad -\sqrt{-1};$$

the last two are imaginary.

3. $x^4 - (2bc + 4a^2)\, x^2 = -b^2c^2.$

Ans. $x = \pm\sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}.$

4. $2x - 7\sqrt{x} = 99.$ *Ans.* $x = 81,\ x = \dfrac{121}{4}$

5. $\dfrac{a}{b} - bx^4 + \dfrac{c}{d}x^2 = 0.$ *Ans.* $x = \pm\sqrt{\dfrac{c \pm \sqrt{4ad^2 + c^2}}{2bd}}$

125. The solution of trinomial equations of the fourth degree requires the extraction of the square root of expressions of the form of $a \pm \sqrt{b}$ in which a and b are positive or negative, numerical or algebraic. The expression $\sqrt{a \pm \sqrt{b}}$ can sometimes be reduced to the form of $a' \pm \sqrt{b'}$ or to the form $\sqrt{a''} \pm \sqrt{b''}$; and when such transformation is possible, it is

advantageous to effect it, since, in this case, we have only to extract two simple square roots; whereas, the expression

$$\sqrt{a \pm \sqrt{b}},$$

requires the extraction of the square root of the square root.

To deduce formulas for making the required transformation, let us assume

$$p + q = \sqrt{a + \sqrt{b}} \quad - \; - \; - \; - \quad (1),$$

$$p - q = \sqrt{a - \sqrt{b}} \quad - \quad (2);$$

in which p and q are arbitrary quantities.

It is now required to find such values for p and q as will satisfy equations (1) and (2).

By squaring both members of equations (1) and (2), we have

$$p^2 + 2pq + q^2 = a + \sqrt{b} \quad - \; - \; - \quad (3),$$

$$p^2 - 2pq + q^2 = a - \sqrt{b} \quad - \; - \; - \quad (4).$$

Adding equations (3) and (4), member to member, we get

$$p^2 + q^2 = a \quad - \; - \; - \; - \; - \; - \; - \quad (5).$$

Multiplying (1) and (2), member by member, we have,

$$p^2 - q^2 = \sqrt{a^2 - b}.$$

Let us now represent $\sqrt{a^2 - b}$ by c. Substituting in the last equation,

$$p^2 - q^2 = c \quad - \; - \; - \; - \; - \; - \; - \quad (6).$$

From (5) and (6) we readily deduce,

$$p = \pm\sqrt{\frac{a+c}{2}} \quad \text{and} \quad q = \pm\sqrt{\frac{a-c}{2}};$$

these values substituted for p and q, in equations (1) and (2), give

$$\sqrt{a + \sqrt{b}} = \pm\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}};$$

$$\sqrt{a - \sqrt{b}} = \pm\sqrt{\frac{a+c}{2}} \mp \sqrt{\frac{a-c}{2}}.$$

hence,

$$\sqrt{a+\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right) \quad . \quad . \quad (7),$$

and

$$\sqrt{a-\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right) \quad . \quad . \quad (8).$$

Now, if a^2-b is a perfect square, its square root, c, will be a rational quantity, and the application of one of the formulas (7) or (8) will reduce the given expression to the required form. If a^2-b is not a perfect square, the application of the formulas will not simplify the given expression, for, we shall still have to extract the square root of a square root.

Therefore, in general, this transformation is not used, unless a^2-b is a perfect square.

EXAMPLES.

1. Reduce $\sqrt{94+42\sqrt{5}}=\sqrt{94+\sqrt{8820}}$, to its simplest form. We have, $a=94, \quad b=8820,$

whence, $c=\sqrt{a^2-b}=\sqrt{8836-8820}=4,$

a rational quantity; formula (7) is therefore applicable to this case, and we have

$$\sqrt{94+42\sqrt{5}}=\pm\left(\sqrt{\frac{94+4}{2}}+\sqrt{\frac{94-4}{2}}\right),$$

or, reducing, $\qquad =\pm(\sqrt{49}+\sqrt{45});$

hence, $\qquad \sqrt{94+42\sqrt{5}}=\pm(7+3\sqrt{5}).$

This may be verified; for,

$$(7+3\sqrt{5})^2=49+45+42\sqrt{5}=94+42\sqrt{5}.$$

2. Reduce $\sqrt{np+2m^2-2m\sqrt{np+m^2}}$, to its simplest form. We have

$$a=np+2m^2, \quad \text{and} \quad b=4m^2(np+m^2),$$

$$a^2-b=n^2p^2, \quad \text{and} \quad c=\sqrt{a^2-b}=np;$$

and therefore, formula (7) is applicable. It gives,

$$\pm\left(\sqrt{\frac{np+2m^2+np}{2}}-\sqrt{\frac{np+2m^2-np}{2}}\right),$$

and, reducing, $\pm\left(\sqrt{np+m^2}-m\right)$.

3. Reduce to its simplest form,

$$\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}.$$

By applying the formulas, we find

$$\sqrt{16+30\sqrt{-1}}=5+3\sqrt{-1},$$

and $$\sqrt{16-30\sqrt{-1}}=5-3\sqrt{-1}:$$

hence, $$\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}=10.$$

This example shows that the transformation is applicable to imaginary expressions.

4. Reduce to its simplest form,

$$\sqrt{28+10\sqrt{3}}. \qquad \textit{Ans. } 5+\sqrt{3}.$$

5. Reduce to its simplest form,

$$\sqrt{1+4\sqrt{-3}}. \qquad \textit{Ans. } 2+\sqrt{-3}.$$

6. Reduce to its simplest form,

$$\sqrt{bc+2b\sqrt{bc-b^2}}-\sqrt{bc-2b\sqrt{bc-b^2}}.$$

Ans. $\pm 2b$

7. Reduce to its simplest form,

$$\sqrt{ab+4c^2-d^2-2\sqrt{4abc^2-abd^2}}.$$

Ans. $\sqrt{ab}-\sqrt{4c^2-d^2}$

Equations of the Second Degree involving two or more unknown quantities.

126. Every equation of the second degree, containing two unknown quantities, is of the general form

$$ay^2 + bxy + cx^2 + dy + fx + g = 0;$$

or a particular case of that form. For, this equation contains terms involving the squares of both unknown quantities, their product, their first powers, and a known term.

In order to discuss, generally, equations of the second degree involving two unknown quantities, let us take the two equations of the most general form

$$a\,y^2 + b\,xy + c\,x^2 + d\,y + f\,x + g\ = 0,$$

and

$$a'y^2 + b'xy + c'x^2 + d'y + f'x + g' = 0.$$

Arranging them with reference to x, they become

$$c\,x^2 + (b\,y + f\,)\,x + a\,y^2 + d\,y + g\ = 0,$$
$$c'x^2 + (b'y + f')\,x + a'y^2 + d'y + g' = 0;$$

from which we may eliminate x^2, after having made its co-efficient the same in both equations.

By multiplying both members of the first equation by c', and both members of the second by c, they become,

$$cc'x^2 + (b\,y + f\,)\,c'x + (a\,y^2 + d\,y + g\,)c' = 0,$$
$$cc'x^2 + (b'y + f')c\,x + (a'y^2 + d'y + g')c\ = 0.$$

Subtracting one from the other, member from member, we have

$$[(bc' - cb')y + fc' - cf']x + (ac' - ca')y^2 + (dc' - cd')y + gc' - cg' = 0,$$

which gives

$$x = \frac{(ca' - ac')y^2 + (cd' - dc')y + cg' - gc'}{(bc' - cb')y + fc' - cf'}.$$

This value being substituted for x in one of the proposed equations, will give *a final equation*, involving only y.

But without effecting the substitution, which would lead to a very complicated result, it is easy to perceive that the final equation involving y, will be of the fourth degree. For, the

numerator of the value of x being of the form

$$my^2 + ny + p,$$

its square will be of the fourth degree, and this square forms one of the parts in the result of the substitution.

Therefore, in general, *the solution of two equations of the second degree, involving two unknown quantities, depends upon that of an equation of the fourth degree, involving one unknown quantity.*

127. Since we have not yet explained the manner of solving equations of the fourth degree, it follows that we cannot, as yet, solve the general case of two equations of the second degree involving two unknown quantities. There are, however, some particular cases that admit of solution, by the application of the rules already demonstrated.

First. We can always solve two equations containing two unknown quantities, when one of the equations is of the second degree, and the other of the first.

For, we can find the value of one of the unknown quantities in terms of the other and known quantities, from the latter equation, and by substituting this in the former, we shall have a single equation of the second degree containing but one unknown quantity, which can be solved.

Thus, if we have the two equations

$$x^2 + 2y^2 = 22 \quad - \; - \; - \quad (1),$$

$$2x - y = 1 \quad - \; - \; - \; - \quad (2),$$

we can find from equation (2),

$$x = \frac{1+y}{2}; \quad \text{whence,} \quad x^2 = \frac{1+2y+y^2}{4};$$

and by substituting this expression for x^2 in equation (1), we find

$$\frac{1+2y+y^2}{4} + 2y^2 = 22;$$

whence we get the values of y: that is,

$$y = 3 \quad \text{and} \quad y = -\frac{29}{9};$$

and by substituting in equation (2) we find,

$$x = 2 \quad \text{and} \quad x = -\frac{10}{9}.$$

Second. We can always solve two equations of the second degree containing two unknown quantities when they are both homogeneous with respect to these quantities.

For, we can substitute for one of the unknown quantities, an auxiliary unknown quantity multiplied into the second unknown quantity, and by combining the two resulting equations we can find an equation of the second degree, from which the value of the auxiliary unknown quantity may be determined, and thence the values of the required quantities can easily be found.

Take, for example, the equations

$$x^2 + xy - y^2 = 5 \quad - \; - \; - \quad (1),$$
$$3x^2 - 2xy - 2y^2 = 6 \quad - \; - \; - \quad (2).$$

Substitute for y, px, p being unknown, the given equations become

$$x^2 + px^2 - p^2x^2 = 5 \quad - \; - \; - \quad (3),$$
$$3x^2 - 2px^2 - 2p^2x^2 = 6 \quad - \; - \; - \quad (4).$$

Finding the values of x^2 in terms of p, from equations (3) and (4), and placing them equal to each other, we deduce

$$\frac{5}{1+p-p^2} = \frac{6}{3-2p-2p^2};$$

or reducing, $$p^2 + 4p = \frac{9}{4};$$

whence, $$p = \frac{1}{2}, \quad \text{and} \quad p = -\frac{9}{2}.$$

Considering the positive value of p, we have, by substituting it in equation (3),

$$x^2\left(1 + \frac{1}{2} - \frac{1}{4}\right) = 5,$$

or, $$x^2 = 4;$$

whence, $$x = 2 \quad \text{and} \quad x = -2:$$

and since $y = px$ we have $y = 1$ and $y = -1$.

Third There are certain other cases which admit of solution, but for which no fixed rule can be given.

We shall illustrate the manner of treating these cases, by the solution of the following

EXAMPLES.

1. Given, $\left.\begin{aligned}\frac{xy}{\sqrt{\frac{x}{y}}} &= 48,\\ \frac{xy}{\sqrt{x}} &= 24,\end{aligned}\right\}$ to find the values of x and y.

Dividing the first by the second, member by member, we have

$$\frac{\sqrt{x}}{\sqrt{\frac{x}{y}}} = 2, \quad \text{or} \quad \sqrt{y} = 2; \quad \text{whence} \quad y = 4;$$

and by substituting in the second equation, we get

$$\sqrt{x} = 6, \quad \text{and} \quad x = 36.$$

2. Given, $\left.\begin{aligned}x + \sqrt{xy} + y &= 19,\\ x^2 + xy + y^2 &= 133,\end{aligned}\right\}$ to find the values of x and y.

Dividing the second by the first, member by member, we have

$$x - \sqrt{xy} + y = 7.$$

But, $$x + \sqrt{xy} + y = 19:$$

adding these, member to member, and dividing by 2, we find

$$x + y = 13,$$

which substituted in the first equation, gives,

$$\sqrt{xy} = 6, \quad \text{or} \quad xy = 36, \quad \text{and} \quad x = \frac{36}{y}.$$

Substituting this expression for x, in the preceding equation, we get,

$$\frac{36}{y} + y = 13,$$

or, $$y^2 - 13y = -36;$$

whence, $$y = \frac{13}{2} \pm \sqrt{-36 + \frac{169}{4}} = \frac{13}{2} \pm \frac{5}{2}:$$

and finally, $$y = 9, \quad \text{or} \quad y = 4;$$

and since $$x + y = 13,$$

$$x = 4, \quad \text{or} \quad x = 9.$$

3. Find the values of x and y, in the equations

$$x^2 + 3x + y = 73 - 2xy$$
$$y^2 + 3y + x = 44.$$

By transposition, the first equation becomes,

$$x^2 + 2xy + 3x + y = 73;$$

to which, if the second be added, member to member, there results,

$$x^2 + 2xy + y^2 + 4x + 4y = (x + y)^2 + 4(x + y) = 117.$$

If, now, in the equation

$$(x + y)^2 + 4(x + y) = 117,$$

we regard $x + y$ as a single unknown quantity, we shall have

$$x + y = -2 \pm \sqrt{117 + 4};$$

hence, $\quad x + y = -2 + 11 = 9,$

and $\quad x + y = -2 - 11 = -13;$

whence, $\quad x = 9 - y, \quad$ and $\quad x = -13 - y.$

Substituting these values of x in the second equation, we have

$$y^2 + 2y = 35, \quad \text{for} \quad x = 9 - y,$$

and $\quad y^2 + 2y = 57, \quad$ for $\quad x = -13 - y.$

The first equation gives,

$$y = 5, \quad \text{and} \quad y = -7,$$

and the second,

$$y = -1 + \sqrt{58}, \quad \text{and} \quad y = -1 - \sqrt{58}.$$

The corresponding values of x, are

$$x = 4, \qquad x = 16;$$

$$x = -12 - \sqrt{58}, \quad \text{and} \quad x = -12 + \sqrt{58}.$$

4. Find the values of x and y, in the equations

$$x^2y^2 + xy^2 + xy = 600 - (y + 2)x^2y^3$$
$$x + y^2 = 14 - y.$$

From the first equation, we have

$$x^2y^2 + (y^2 + 2y)x^2y^2 + xy^2 + xy = 600,$$

or, $\quad x^2y^2(1 + y^2 + 2y) + xy(1 + y) = 600,$

or, again, $\quad x^2y^2(1 + y)^2 + xy(1 + y) = 600;$

which is of the form of an equation of the second degree, regarding $xy(1+y)$ as the unknown quantity. Hence,

$$xy(1+y) = -\tfrac{1}{2} \pm \sqrt{600+\tfrac{1}{4}} = -\tfrac{1}{2} \pm \sqrt{\frac{2401}{4}};$$

and if we discuss only the roots which belong to the + value of the radical, we have

$$xy(1+y) = -\frac{1}{2} + \frac{49}{2} = 24;$$

and hence, $$x = \frac{24}{y+y^2}.$$

Substituting this value for x in the second equation, we have

$$(y^2+y)^2 - 14(y^2+y) = -24;$$

whence, $y^2+y=12$, and $y^2+y=2$.

From the first equation, we have

$$y = -\frac{1}{2} \pm \frac{7}{2} = 3, \quad \text{or} \quad -4;$$

and the corresponding values of x, from the equation

$$x = \frac{24}{y^2+y} = 2.$$

From the second equation, we have

$$y = 1, \quad \text{and} \quad y = -2;$$

which gives $x = 12$.

5. Given, $x^2y + xy^2 = 6$, and $x^3y^2 + x^2y^3 = 12$, to find the values of x and y.

Ans. $\begin{cases} x = 2 \quad \text{or} \quad 1, \\ y = 1 \quad \text{or} \quad 2. \end{cases}$

6. Given, $\begin{Bmatrix} x^2+x+y = 18-y^2 \\ xy = 6 \end{Bmatrix}$ to find the values of x and y.

Ans. $\begin{cases} x = 3, \text{ or } 2; \text{ or } -3 \pm \sqrt{3}, \\ y = 2, \text{ or } 3; \text{ or } -3 \mp \sqrt{3}. \end{cases}$

Problems giving rise to Equations of the Second Degree containing two or more unknown quantities.

1. Find two numbers such, that the sum of the respective products of the first multiplied by a, and the second multiplied by b, shall be equal to $2s$; and the product of the one by the other equal to p.

Let x and y denote the required numbers, and we have

$$ax + by = 2s,$$

and

$$xy = p.$$

From the first

$$y = \frac{2s - ax}{b};$$

whence, by substituting in the second, and reducing,

$$ax^2 - 2sx = -bp.$$

Therefore,

$$x = \frac{s}{a} \pm \frac{1}{a}\sqrt{s^2 - abp},$$

and consequently,

$$y = \frac{s}{b} \mp \frac{1}{b}\sqrt{s^2 - abp}.$$

Let $a = b = 1$; the values of x, and y, then reduce to

$$x = s \pm \sqrt{s^2 - p}, \quad \text{and} \quad y = s \mp \sqrt{s^2 - p};$$

whence we see that, under this supposition, the two values of x are equal to those of y, taken in an inverse order; which shows, that if

$s + \sqrt{s^2 - p}$ represents the value of x, $s - \sqrt{s^2 - p}$

will represent the corresponding value of y, and conversely.

This relation is explained by observing that, under the last supposition, the given equations become

$$x + y = 2s, \quad \text{and} \quad xy = p;$$

and the question is then reduced to *finding two numbers of which the sum is* $2s$, *and their product* p; or in other words, *to divide a number* $2s$, *into two such parts, that their product may be equal to a given number* p.

2. To find four numbers, such that the sum of the first and fourth shall be equal to $2s$, the sum of the second and third equal to $2s'$, the sum of their squares equal to $4c^2$, and the product of the first and fourth equal to the product of the second and third.

Let u, x, y, and z, denote the numbers, respectively. Then, from the conditions of the problem, we shall have

$$\begin{array}{rll} u + z = 2s & \text{1st condition;} \\ x + y = 2s' & \text{2d} & \text{"} \\ u^2 + x^2 + y^2 + z^2 = 4c^2 & \text{3d} & \text{"} \\ uz = xy & \text{4th} & \text{"} \end{array}$$

At first sight, it may appear difficult to find the values of the unknown quantities, but by the aid of an *auxiliary unknown quantity*, they are easily determined.

Let p be the unknown product of the 1st and 4th, or 2d and 3d; we shall then have

$$\left\{\begin{array}{r} u + z = 2s, \\ uz = p, \end{array}\right\} \text{ which give, } \left\{\begin{array}{l} u = s + \sqrt{s^2 - p}, \\ z = s - \sqrt{s^2 - p}. \end{array}\right.$$

and

$$\left\{\begin{array}{r} x + y = 2s', \\ xy = p, \end{array}\right\} \text{ which give, } \left\{\begin{array}{l} x = s' + \sqrt{s'^2 - p}, \\ y = s' - \sqrt{s'^2 - p}. \end{array}\right.$$

Now, by substituting these values of u, x, y, z, in the third equation of the problem, it becomes

$$(s + \sqrt{s^2 - p})^2 + (s - \sqrt{s^2 - p})^2 + (s' + \sqrt{s'^2 - p})^2$$
$$+ (s' - \sqrt{s'^2 - p})^2 = 4c^2;$$

and by developing and reducing,

$$4s^2 + 4s'^2 - 4p = 4c^2; \quad \text{hence,} \quad p = s^2 + s'^2 - c^2.$$

Substituting this value for p, in the expressions for u, x, y, z, we find

$$\left\{\begin{array}{l} u = s + \sqrt{c^2 - s'^2}, \\ z = s - \sqrt{c^2 - s'^2}, \end{array}\right. \qquad \left\{\begin{array}{l} x = s' + \sqrt{c^2 - s^2}, \\ y = s' - \sqrt{c^2 - s^2}. \end{array}\right.$$

These values evidently satisfy the last equation of the problem; for

$$uz = (s + \sqrt{c^2 - s'^2})(s - \sqrt{c^2 - s'^2}) = s^2 - c^2 + s'^2,$$

$$xy = (s' + \sqrt{c^2 - s^2})(s' - \sqrt{c^2 - s^2}) = s'^2 - c^2 + s^2.$$

REMARK.—This problem shows how much the introduction of an *unknown auxiliary* often facilitates the determination of the principal unknown quantities. There are other problems of the same kind, which lead to equations of a degree superior to the second, and yet they may be resolved by the aid of equations of the first and second degrees, by introducing *unknown auxiliaries.*

3. Given the sum of two numbers equal to a, and the sum of their cubes equal to c, to find the numbers

By the conditions $\begin{cases} x + y = a \\ x^3 + y^3 = c. \end{cases}$

Putting $x = s + z$, and $y = s - z$, we have $a = 2s$,

and $\begin{cases} x^3 = s^3 + 3s^2z + 3sz^2 + z^3 \\ y^3 = s^3 - 3s^2z + 3sz^2 - z^3: \end{cases}$

hence, by addition, $x^3 + y^3 = 2s^3 + 6sz^2 = c;$

whence, $z^2 = \dfrac{c - 2s^3}{6s}$, and $z = \pm\sqrt{\dfrac{c - 2s^3}{6s}}$,

or, $x = s \pm\sqrt{\dfrac{c - 2s^3}{6s}}$, and $y = s \mp\sqrt{\dfrac{c - 2s^3}{6s}}$;

and by substituting for s its value,

$$x = \frac{a}{2} \pm\sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \pm\sqrt{\frac{4c - a^3}{12a}},$$

and $$y = \frac{a}{2} \mp\sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \mp\sqrt{\frac{4c - a^3}{12a}}.$$

4. The sum of the squares of two numbers is expressed by a, and the difference of their squares by b: what are the numbers?

Ans. $\sqrt{\dfrac{a + b}{2}}$, $\sqrt{\dfrac{a - b}{2}}$.

5. What three numbers are they, which, multiplied two and two, and each product divided by the third number, give the quotients, a, b, c?

Ans. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

6. The sum of two numbers is 8, and the sum of their cubes is 152: what are the numbers? *Ans.* 3 and 5.

7. Find two numbers, whose difference added to the difference of their squares is 150, and whose sum added to the sum of their squares, is 330. *Ans.* 9 and 15.

8. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number: what are the numbers? *Ans.* 3 and 18.

9. What two numbers are those whose sum multiplied by the greater, is equal to 77; and whose difference, multiplied by the lesser, is equal to 12?

Ans. 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

10. Divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

11. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. *Ans.* 10 and 14.

12. What two numbers are they, whose product is 255, and the sum of whose squares is 514? *Ans.* 15 and 17.

13. There is a number expressed by two digits, which, when divided by the sum of the digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by 1 than the sum of the digits, the quotient will exceed the former quotient by 2: what is the number? *Ans.* 24.

14. A regiment, in garrison, consisting of a certain number of companies, receives orders to send 216 men on duty, each company to furnish an equal number. Before the order was executed, three of the companies were sent on another service, and it was then found that each company that remained would have to send 12 men additional, in order to make up the complement, 216. How many companies were in the regiment, and what number of men did each of the remaining companies send

Ans. 9 companies: each that remained sent 36 men.

15. Find three numbers such, that their sum shall be 14, the sum of their squares equal to 84, and the product of the first and third equal to the square of the second.

Ans. 2, 4 and 8.

16. It is required to find a number, expressed by three digits, such, that the sum of the squares of the digits shall be 104; the square of the middle digit to exceed twice the product of the other two by 4; and if 594 be subtracted from the number, the remainder will be expressed by the same figures, but with the extreme digits reversed. *Ans.* 862.

17. A person has three kinds of goods which together cost $\$230\frac{5}{24}$. A pound of each article costs as many $\frac{1}{24}$ dollars as there are pounds in that article: he has one-third more of the second than of the first, and $3\frac{1}{2}$ times as much of the third as of the second: How many pounds has he of each article?

Ans. 15 of the 1st, 20 of the 2d, 70 of the 3d.

18. Two merchants each sold the same kind of stuff: the second sold 3 yards more of it than the first, and together, they received 35 dollars. The first said to the second, "I would have received 24 dollars for your stuff." The other replied, "And I would have received $12\frac{1}{2}$ dollars for yours." How many yards did each of them sell?

Ans. $\left\{\begin{array}{ll}\text{1st merchant} & 15\\ \text{2d - - -} & 18\end{array}\right\}$ or $\left\{\begin{array}{l}5\\ 8.\end{array}\right.$

19. A widow possessed 13000 dollars, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Ans. 7 and 6 per cent.

CHAPTER VII.

FORMATION OF POWERS—BINOMIAL THEOREM—EXTRACTION OF ROOTS OF ANY DEGREE—OF RADICALS.

128. THE solution of equations of the second degree supposes the process for extracting the square root to be known. In like manner, the solution of equations of the third, fourth, &c., degrees, requires that we should know how to extract the third, fourth, &c., roots of any numerical or algebraic quantity.

The power of a number can be obtained by the rules for multiplication, and this power is subject to a certain *law of formation*, which it is necessary to know, in order to *deduce the root from the power.*

Now, the law of formation of the square of a numerical or algebraic quantity, is deduced from the expression for the square of a binomial (Art. 47); so likewise, the law of a power of any degree, is deduced from the expression for the same power of a binomial. We shall therefore first determine *the law for the formation of any power of a binomial.*

129. By taking the binomial $x + a$ several times, as a factor, the following results are obtained, by the rule for multiplication

$$
\begin{aligned}
(x+a) &= x + a,\\
(x+a)^2 &= x^2 + 2ax + a^2,\\
(x+a)^3 &= x^3 + 3ax^2 + 3a^2x + a^3,\\
(x+a)^4 &= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,\\
(x+a)^5 &= x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.
\end{aligned}
$$

By examining these powers of $x + a$, we readily discover *the law* according to which the exponents of the powers of a de

crease, and those of the powers of a increase, in the successive terms. It is not, however, so easy to discover a law for the formation of the co-efficients. Newton discovered one, by means of which a binomial may be raised to any power, without performing the multiplications. He did not, however, explain the course of reasoning which led him to the discovery; but the law has since been demonstrated in a rigorous manner. Of all the known demonstrations of it, the most elementary is that which is founded upon the *theory of combinations.* However, as the demonstration is rather complicated, we will, in order to simplify it, begin by demonstrating some propositions relative to permutations and combinations, on which the demonstration of the binomial theorem depends.

Of Permutations, Arrangements and Combinations.

130. Let it be proposed to determine the *whole number of ways* in which several letters, a, b, c, d, &c., can be written one after the other. The result corresponding to each change in the position of any one of these letters, is called a *permutation.*

Thus, the two letters a and b furnish the two *permutations,* ab and ba.

In like manner, the three letters, a, b, c, furnish six permutations.

$$\begin{cases} cab \\ acb \\ abc \\ cba \\ bca \\ bac \end{cases}$$

PERMUTATIONS, *are the results obtained by writing a certain number of letters one after the other, in every possible order, in such a manner that all the letters shall enter into each result, and each letter enter but once.*

To determine the number of permutations of which n letters are susceptible.

Two letters, a and b, evidently give two permutations.

$$\begin{cases} ab \\ ba \end{cases}$$

Therefore, the number of permutations of two letters is expressed by 1×2.

Take the three letters, a, b, and c. Reserve either of the letters, as c, and permute the other two, giving

$$c \left\{ \begin{array}{l} ab \\ ba \end{array} \right.$$

Now, the third letter c may be placed before ab, between a and b, and at the right of ab; and the same for ba: that is, in ONE *of the first permutations, the reserved letter* c *may have three different places, giving three permutations.* And, as the same may be shown for *each* one of the first permutations, it follows that the whole number of permutations of three letters will be expressed by, $1 \times 2 \times 3$.

$$\left\{ \begin{array}{l} cab \\ acb \\ abc \\ cba \\ bca \\ bac \end{array} \right.$$

If, now, a fourth letter d be introduced, it can have four places in *each one* of the six permutations of three letters: hence, the number of permutations of four letters will be expressed by, $1 \times 2 \times 3 \times 4$.

In general, let there be n letters, a, b, c, &c., and suppose the total number of permutations of $n-1$ letters to be known; and let Q denote that number. Now, in *each one* of the Q permutations, the reserved letter may have n places, giving n permutations: hence, when it is so placed in all of them, the entire number of permutations will be expressed by $Q \times n$.

If $n = 5$, Q will denote the number of permutations of four quantities, or will be equal to $1 \times 2 \times 3 \times 4$; hence, the number of permutations of five quantities will be expressed by $1 \times 2 \times 3 \times 4 \times 5$.

If $n = 6$, we shall have for the number of permutations of six quantities, $1 \times 2 \times 3 \times 4 \times 5 \times 6$, and so on.

Hence, if Y denote the number of permutations of n letters, we shall have

$$Y = Q \times n = 1. \quad 2. \quad 3. \quad 4. \quad . \quad . \quad . \quad (n-1)n: \text{ that is,}$$

The number of permutations of n letters, is equal to the continued product of the natural numbers from 1 *to n inclusively.*

Arrangements.

131. Suppose we have a number m, of letters a, b, c, d, &c. If they are written in sets of 2 and 2, or 3 and 3, or 4 and 4 . . . in every possible order in each set, such results are called *arrangements.*

Thus, ab, ac, ad, . . . ba, bc, bd, . . . ca, cb, cd, . . . are *arrangements* of m letters taken 2 and 2; or in sets of 2 each.

In like manner, abc, abd, . . . bac, bad, . . . acb, acd, . . . are *arrangements* taken in sets of 3.

ARRANGEMENTS, *are the results obtained by writing a number m of letters, in sets of* 2 *and* 2; 3 *and* 3, 4 *and* 4, . . . *n and n; the letters in each set having every possible order, and m being always greater than n.*

If we suppose $m = n$, the *arrangements*, taken n and n, become *permutations.*

Having given a number m of letters a, b, c, d, . . . to determine the total number of arrangements that may be formed of them by taking them n in a set.

Let it be proposed, in the first place, to arrange three letters, a, b and c, in sets of two each.

First, arrange the letters in sets of one each, and for each set so formed, there will be two letters reserved: the reserved letters for either arrangement, being those which do not enter it. Thus, with reference to a, the reserved letters are b and c; with reference to b, the reserved letters are a and c; and with reference to c, they are a and b.

$$\begin{cases} a \\ b \\ c \end{cases}$$

Now, to any one of the letters, as a, annex, in succession, the reserved letters b and c: to the second arrangement b, annex the reserved letters a and c and to the third arrangement, c, annex the reserved letters a and b.

$$\begin{cases} a\,b \\ a\,c \\ b\,a \\ b\,c \\ c\,a \\ c\,b \end{cases}$$

Since each of the first arrangements gives as many new arrangements as there are reserved letters, it follows, *that the*

number of arrangements of three letters taken, two in a set, will be equal to the number of arrangements of the same letters taken one in a set, multiplied by the number of reserved letters.

Let it be required to form the arrangement of four letters, a, b, c and d; taken three in a set.

First, arrange the four letters in sets of two: there will then be for each arrangement, two reserved letters. Take *one* of the sets and write after it, in succession, each of the reserved letters: we shall thus form as many sets of three letters each as there are reserved letters; and these sets differ from each other by at least the last letter. Take another of the first arrangements, and annex, in succession, the reserved letters; we shall again form as many different arrangements as there are reserved letters. Do the same for all of the first arrangements, and it is plain, that the whole number of arrangements which will be formed, of four letters, taken 3 and 3, *will be equal to the number of arrangements of the same letters, taken two in a set, multiplied by the number of reserved letters.*

$$\text{In sets of two.}\left\{\begin{array}{l} ab \\ ac \\ ad \\ ba \\ bc \\ bd \\ ca \\ cb \\ cd \\ da \\ db \\ dc \end{array}\right.$$

In general, suppose the total number of *arrangements* of m letters, taken $n-1$ in a set, to be known, and denote this number by P.

Take any one of these arrangements, and annex to it, in succession, each of the reserved letters, of which the number is $m-(n-1)$, or $m-n+1$. It is evident, that we shall thus form a number $m-n+1$ of new arrangements of n letters, each differing from the others by the last letter.

Now, take another of the first arrangements of $n-1$ letters, and annex to it, in succession, each of the $m-n+1$ letters which do not enter it; we again obtain a number $m-n+1$ of arrangements of n letters, differing from each other, and from those obtained as above, by at least one of the $n-1$ first letters. Now, as we may in the same manner, take all the P arrangements of the m letters, taken $n-1$ in a set, and annex to them

in succession, each of the $m - n + 1$ other letters, it follows that the total number of arrangements of m letters, taken n in a set, is expressed by

$$P(m - n + 1).$$

To apply this, in determining the number of arrangements of m letters, taken 2 and 2, 3 and 3, 4 and 4, or 5 and 5 in a set, make $n = 2$; whence, $m - n + 1 = m - 1$; P in this case, will express the total number of arrangements, taken $2 - 1$ and $2 - 1$, or 1 and 1; and is consequently equal to m; therefore, the expression

$$P(m - n + 1) \text{ becomes } m(m - 1).$$

Let $n = 3$; whence, $m - n + 1 = m - 2$; P will then express the number of arrangements taken 2 and 2, and is equal to $m(m - 1)$; therefore, the expression becomes

$$m(m - 1)(m - 2).$$

Again, take $n = 4$: whence, $m - n + 1 = m - 3$: P will express the number of arrangements taken 3 and 3, and therefore the expression becomes

$$m(m - 1)(m - 2)(m - 3), \text{ and so on.}$$

Hence, if we denote the number of arrangements of m letters, taken n in a set by X, we shall have,

$$X = P(m - n + 1) = m(m - 1)(m - 2) \,.\,.\, (m - n + 1); \text{ that is,}$$

The number of arrangements of m letters, taken n in a set, is equal to the continued product of the natural numbers from m down to $m - n + 1$, *inclusively.*

If in the preceding formula m be made equal to n, the arrangements become permutations, and the formula reduces to

$$X = n(n - 1)(n - 2) \,.\,.\,.\,.\, 2 \,.\, 1;$$

or, by reversing the order of the factors, and writing Y for X,

$$Y = 1 \,.\, 2 \,.\, 3 \,.\,.\,.\,.\, (n - 1)n;$$

the same formula as deduced in the last article.

Combinations.

132. When the letters are disposed, as in the arrangements, 2 and 2, 3 and 3, 4 and 4, &c., and it is required that any two of the results, thus formed, shall differ by at least one letter. the *products* of the letters will be different. In this case, the results are called *combinations.*

Thus. ab, ac, bc, . . . ad, bd, . . . are *combinations* of the letters a, b, c, and d, &c., taken 2 and 2.

In like manner, abc, abd, . . . acd, bcd, . . . are *combinations* of the letters taken 3 and 3: hence,

COMBINATIONS, *are arrangements in which any two will differ from each other by at least one of the letters which enter them.*

To determine the total number of different combinations that can be formed of m letters, taken n in a set.

Let X denote the total number of arrangements that can be formed of m letters, taken n and n: Y the number of per mutations of n letters, and Z the total number of *different combinations* taken n and n.

It is evident, that all the possible arrangements of m letters taken n in a set, can be obtained, by subjecting the n letters of each of the Z combinations, to all the *permutations* of which these letters are susceptible. Now, a single combination of n letters gives, by hypothesis, Y permutations or arrangements: therefore Z combinations will give $Y \times Z$ arrangements; and as X denotes the total number of arrangements, it follows that

$$X = Y \times Z; \quad \text{whence,} \quad Z = \frac{X}{Y}.$$

But we have (Art. 130),

$$Y = Q \times n = 1 \;.\; 2 \;.\; 3 \;.\; .\; .\; .\; .\; n,$$

and (Art. 131),

$$X = P(m - n + 1) = m(m-1)(m-2) \;.\;.\;.\;.\; (m - n + 1);$$

therefore,

$$Z = \frac{P(m-n+1)}{Q \times n} = \frac{m(m-1)(m-2) \;.\;.\;.\;.\; (m-n+1)}{1 \;.\; 2 \;.\; 3 \;.\;.\;.\;.\;.\;.\;.\; n};$$

that is,

The number of combinations of m letters taken n in a set, is equal to the continued product of the natural numbers from m down to $m-n+1$ inclusively, divided by the continued product of the natural numbers from 1 to n inclusively.

133. If Z denote the number of combinations of the m letters taken n in a set, we have just seen that

$$Z = \frac{m(m-1)(m-2)\ldots.(m-n+1)}{1.2.3\ldots\ldots n} \quad \ldots. (1).$$

If Z' denote the number of combinations of m letters taken $(m-n)$ in a set, we can find an expression for Z' by changing n into $m-n$ in the second member of the above formula; whence

$$Z' = \frac{m(m-1)(m-2)\ldots.(n+1)}{1.2.3\ldots\ldots(m-n)} \quad \ldots\ldots (2).$$

If, now, we divide equation (1) by (2), member by member, and arrange the factors of both terms of the quotient, we shall have

$$\frac{Z}{Z'} = \frac{1.2.3\ldots.(m-n)\times(m-n+1)\ \ldots\ (m-1)m}{1.2.3\ldots.\ n\times(n+1)\ \ldots\ldots\ (m-1)m}.$$

The numerator and denominator of the second member are equal to each other, since each contains the factors, 1, 2, 3, &c., to m; hence,

$$\frac{Z}{Z'} = 1, \quad \text{or} \quad Z = Z'; \quad \text{therefore,}$$

The number of combinations of m letters, taken n in a set, is equal to the number of combinations of m letters, taken $m-n$ in a set.

Binomial Theorem.

134. The object of this theorem is to show how to find any power of a binomial, without going through the process of con tinued multiplication.

135. The algebraic equation which indicates the law of formation of any power of a binomial, is called the *Binomial Formula.*

In order to discover this law for the mth power of the binomial $x + a$, let us observe the law for the formation of the product of several binomial factors, $x + a$, $x + b$, $x + c$, $x + d$ . . of which the first term is the same in all, and the second terms different.

$$\begin{array}{l|l|l|l|l}
 & x + a & & & \\
 & x + b & & & \\
\hline
\text{1st product} & x^2 + a & x + ab & & \\
 & \quad + b & & & \\
 & x + c & & & \\
\hline
\text{2d} \;.\;.\;.\;.\;. & x^3 + a & x^2 + ab & x + abc & \\
 & \quad + b & \quad + ac & & \\
 & \quad + c & \quad + bc & & \\
 & x + d & & & \\
\hline
\text{3d} \;.\;.\;.\;.\;. & x^4 + a & x^3 + ab & x^2 + abc & x + abcd \\
 & \quad + b & \quad + ac & \quad + abd & \\
 & \quad + c & \quad + ad & \quad + acd & \\
 & \quad + d & \quad + bc & \quad + bcd & \\
 & & \quad + bd & & \\
 & & \quad + cd & &
\end{array}$$

These products, obtained by the common rule for algebraic multiplication, indicate the following laws:—

1st. With respect to the exponents, we observe that the exponent of x, in the first term, is equal to the number of binomial factors employed. In each of the following terms to the right, this exponent is diminished by 1 to the last term, where it is 0.

2d. With respect to the co-efficients of the different powers of x, that of the first term is 1; the co-efficient of the second term is equal to the sum of the second terms of the binomials; the co-efficient of the third term is equal to the sum of the products of the different second terms, taken two and two;

the co-efficient of the fourth term is equal to the sum of their different products, taken three and three.

Reasoning from *analogy*, we might conclude that, in the product of any number of binomial factors, the co-efficient of the term which has n terms before it, is equal to the sum of the different products of the second terms of the binomials, taken n and n. The last term of the product is equal to the continued product of the second terms of the binomials.

In order to prove that this law of formation is general, suppose that it has been proved true for the product of m binomials. Let us see if it will continue to be true when the product is multiplied by a new binomial factor of the same form.

For this purpose, suppose

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \ldots + Mx^{m-n+1} + Nx^{m-n} + \ldots + U$$

to be the product of m binomial factors; Nx^{m-n} representing the term which has n terms before it, and Mx^{m-n+1} the term which immediately precedes.

Let $x + k$ be the new binomial factor by which we multiply; the product, when arranged according to the powers of x, will be

$$\begin{array}{r|r|r|r|r} x^{m+1}+A & x^m+B & x^{m-1}+C & x^{m-2}+\ldots+N & x^{m-n+1}+\ldots \\ +k & +Ak & +Bk & +Mk & +Uk; \end{array}$$

from which we perceive that the *law of the exponents* is evidently the same.

With respect to the co-efficients, we observe;

1st. That the co-efficient of the first term is 1; and

2d. That $A + k$, or the co-efficient of x^m, is the *sum of the second terms of the* $m + 1$ *binomials.*

3d. Since, by hypothesis, B is the sum of the different products of the second terms of the m binomials, taken two and two, and since $A \times k$ expresses the sum of the products of each of the second terms of the first m binomials by the new second term k; therefore, $B + Ak$ *is the sum of the different products of the second terms of the* $m + 1$ *binomials, taken two and two.*

In general, since N expresses the sum of the products of the second terms of the m binomials, taken n and n, and M the sum of their products, taken $n-1$ and $n-1$, therefore $N+Mk$, or the co-efficient of the term which has n terms before it, will be equal to the sum of the different products of the second terms of the $m+1$ binomials, taken n and n. The last term is equal to the continued product of the second terms of the $m+1$ binomials.

Hence, the law of composition, supposed true for a number m of binomial factors, is also true for a number denoted by $m+1$.

But we have shown the law of composition for 4 factors, hence, the same law is true for 5; and being true for 5, it must be for 6, and so on; hence, it is general.

136. Let us take the equation,

$$(x+a)(x+b)(x+c) \ldots\ldots = x^m + Ax^{m-1} + Bx^{m-2} \ldots\ldots + Nx^{m-n} \ldots\ldots + W,$$

containing in the first member, m binomial factors. If we make

$$a = b = c = d \ldots\ldots \&c.,$$

the first member becomes,

$$(x+a)^m.$$

In the second member the co-efficient of x^m will still be 1. The co-efficient of x^{m-1}, being $a+b+c+d, \ldots$ will become a taken m times; that is, ma.

The co-efficient of x^{m-2}, being

$$ab + ac + ad \ldots\ldots \text{ reduces to } a^2 + a^2 + a^2 \ldots$$

that is, it becomes a^2 taken as many times as there are combinations of m letters, taken two and two, and hence reduces (Art. 132), to

$$m \cdot \frac{m-1}{2} a^2.$$

The co-efficient of x^{m-3} reduces to the product of a^3, multiplied by the number of different combinations of m letters taken three and three; that is, to

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^3, \ \&c.$$

Let us denote the general term, that is, the one which has n terms before it, by Nx^{m-n}.

Then, the co-efficient N will denote the sum of the products of the second terms, taken n and n; and when all the second terms are supposed equal, it becomes equal to a^n multiplied by the number of combinations of m letters, taken n and n. Therefore, the co-efficient of the general term (Art. 132), is

$$\frac{P(m-n+1)}{Q \times n} a^n;$$

hence, we have, by making these substitutions,

$$(x+a)^m = x^m + max^{m-1} + m \cdot \frac{m-1}{2} a^2x^{m-2}$$

$$+ m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} a^3x^{m-3} \ldots + \frac{P(m-n+1)}{Q \cdot n} a^n x^{m-n} \ \ldots + a^m,$$

which is the *binomial formula.*

The term

$$\frac{P(m-n+1)}{Qn} a^n x^{m-n}$$

is called *the general term*, because by making $n = 2, 3, 4,$ &c., all the others can be deduced from it. The term which immediately precedes it, is

$$\frac{P}{Q} a^{n-1} x^{m-n+1}, \quad \text{since} \quad \frac{P}{Q}$$

evidently expresses the number of combinations of m letters taken $n-1$ and $n-1$. Hence, we see, that

$$\frac{P(m-n+1)}{Q \times n},$$

which is called the numerical co-efficient of the general term, is equal to the numerical co-efficient $\frac{P}{Q}$ of the preceding term, multiplied by $m-n+1$, the exponent of x in that term, and divided by n, the number of terms preceding the required term.

The *simple law*, demonstrated above, enables us to determine the numerical co-efficient of any term from that of the preceding term, by means of the following

RULE.

The numerical co-efficient of any term after the first, is formed by multiplying that of the preceding term by the exponent of x *in that term, and dividing the product by the number of terms which precede the required term.*

137. Let it be required to develop

$$(x + a)^6.$$

By applying the foregoing principles, we find,

$$(x + a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

Having written the first term x^6, and the literal parts of the other terms, we find the numerical co-efficient of the second term by multiplying 1, the numerical co-efficient of the first term, by 6, the exponent of x in that term, and dividing by 1, the number of terms preceding the required term. To obtain the co-efficient of the third term, multiply 6 by 5 and divide the product by 2; we get 15 for the required number. The other numerical co-efficients may be found in the same manner

In like manner, we find

$$(x + a)^{10} = x^{10} + 10ax^9 + 45a^2x^8 + 120a^3x^7 + 210a^4x^6$$
$$+ 252a^5x^5 + 210a^6x^4 + 120a^7x^3 + 45a^8x^2 + 10a^9x + a^{10}.$$

138. The operation of finding the numerical co-efficients may be much simplified by the aid of the following principle.

We have seen that the development of $(x + a)^m$, contains $m + 1$ terms; consequently, the term which has n terms after it, has $m - n$ terms before it. Now, the numerical co-efficient of the term which has n terms before it is equal to the number of combinations of m letters taken n in a set, and the numerical co-efficient of that term which has n terms after it, or $m - n$ before it, is equal to the number of combinations of m letters taken $m - n$ in a set; but we have shown (Art. 133) that these numbers are equal. Hence,

In the development of any power of a binomial of the form $(x + a)^m$, *the numerical co-efficients of terms at equal distances from the two extremes, are equal to each other.*

We see that this is the case in both of the examples above given. In finding the development of any power of a binomial, we need find but half, or one more than half, of the numerical co-efficients, since the remaining ones may be written directly from those already found.

139. It frequently happens that the terms of the binomial, to which the formula is to be applied, contain co-efficients and exponents, as in the following example.

Let it be required to raise the binomial

$$3a^2c - 2bd$$

to the fourth power.

Placing $\qquad 3a^2c = x \qquad$ and $\qquad -2bd = y, \qquad$ we have

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4;$$

and substituting for x and y their values, we have

$$(3a^2c - 2bd)^4 = (3a^2c)^4 + 4(3a^2c)^3(-2bd) + 6(3a^2c)^2(-2bd)^2 \\ + 4(3a^2c)(-2bd)^3 + (-2bd)^4,$$

or, by performing the operations indicated,

$$(3a^2c - 2bd)^4 = 81a^8c^4 - 216a^6c^3bd + 216a^4c^2b^2d^2 - 96a^2cb^3d^3 \\ + 16b^4d^4.$$

The terms of the development are alternately plus and minus, as they should be, since the second term is $-$.

140. A power of any polynomial may easily be found by means of the binomial formula, as in the following example.

Let it be required to find the third power of

$$a + b + c.$$

First, put $\qquad b + c = d.$

Then $\quad (a + b + c)^3 = (a + d)^3 = a^3 + 3a^2d + 3ad^2 + d^3,$

and by substituting for the value of d,

$$\begin{aligned}(a + b + c)^3 = a^3 + 3a^2b &+ 3ab^2 + b^3 \\ 3a^2c &+ 3b^2c + 6abc \\ &+ 3ac^2 + 3bc^2 \\ &\qquad + c^3.\end{aligned}$$

This development is composed of *the sum of the cubes of the three terms*, plus *the sum of the results obtained by multiplying three times the square of each term, by each of the other terms in succession*, plus *six times the product of the three terms.*

To apply the preceding formula to the development of the cube of a trinomial, in which the terms are affected with coefficients and exponents, *designate each term by a single letter, and perform the operations indicated; then replace the letters introduced, by their values.*

From this rule, we find that

$$(2a^2 - 4ab + 3b^2)^3 = 8a^6 - 48a^5b + 132a^4b^2 - 208a^3b^3$$
$$+ 198a^2b^4 - 108ab^5 + 27b^6.$$

The fourth, fifth, &c., powers of any polynomial can be developed in a similar manner.

Extraction of the Cube Root of Numbers.

141. The *cube root* of a number, is such a number as being taken three times as a factor, will produce the given number.

A number whose cube root can be exactly found, is called a perfect cube; all other numbers are *imperfect cubes.*

The first ten numbers are,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their cubes,

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Conversely, the numbers in the first line are the cube roots of the corresponding numbers in the second.

If we wish to find the cube root of any number less than 1000, we look for the number in the second line, and if it is there written, the corresponding number in the first line will be its cube root. If the number is not there written, it will fall between two numbers in the second line, and its cube root will fall between the corresponding numbers in the first line. In this case the cube root cannot be expressed in exact parts of 1; hence, the given number must be an imperfect cube (Remark III, Art. 95).

If the given number is greater than 1000, its cube root will be greater than 10; that is, it will contain a certain number of tens and a certain number of units.

Let us designate any number by N, and denote its tens by a, and its units by b; we shall have,

$N = a + b$; whence, $N^3 = a^3 + 3a^2b + 3ab^2 + b^3$; that is,

The cube of a number is equal to the cube of the tens, plus three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.

Thus $(47)^3 = (40)^3 + 3 \times (40)^2 \times 7 + 3 \times 40 \times (7)^2 + (7)^3 = 103823$.

Let us now reverse the operation, and find the cube root of 103823.

$$
\begin{array}{r|l|}
 & 10\dot{3}\ 82\dot{3} & 47 \\
 & 64 & \overline{8} \\
\cline{2-2}
4^2 \times 3 = 48 & 398'23 &
\end{array}
\qquad\qquad
\begin{array}{r}
48 \\
48 \\
\hline
384 \\
192 \\
\hline
2304 \\
48 \\
\hline
18432 \\
9216 \\
\hline
110592
\end{array}
\qquad\qquad
\begin{array}{r}
47 \\
47 \\
\hline
329 \\
188 \\
\hline
2209 \\
47 \\
\hline
15463 \\
8836 \\
\hline
103823
\end{array}
$$

Since the number is greater than 1000, its root will contain tens and units. We will first find the number of tens in the root. Now the cube of tens, giving at least thousands, we point off three places of figures on the right, and the cube of the number of tens will be found in the number 103, to the left of this period.

The cube root of the greatest cube contained in 103 being 4, this is the number of tens in the required root. Indeed, 103823 is evidently comprised between $(40)^3$ or 64,000, and $(50)^3$ or 125,000; hence, the required root is comprised between 4 tens and 5 tens: that is, it is composed of 4 tens, plus a certain number of units less than *ten*.

Having found the number of tens, subtract its cube, 64, from 103, and there remains 39, to which bring down the part 823, and we have 39823, which contains *three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.*

Now, as the square of tens gives at least hundreds, it follows that the product of three times the square of the tens by the units, must be found in the part 398, to the left of 23, which is separated from it by a dash. Therefore, dividing 398 by 48, which is three times the square of the tens, the quotient 8 will be the units of the root, or something greater, since 398 is composed of three times the square of the tens by the units, and generally contains numbers coming from the two other parts.

We may ascertain whether the figure 8 is too great, by forming from the 4 tens and 8 units, the three parts which enter into 39823; but it is much easier to cube 48, as has been done in the above table. Now, the cube of 48 is 110592, which is greater than 103823; therefore, 8 is too great. By cubing 47, we obtain 103823; hence the proposed number is a perfect cube, and 47 is its cube root.

By a course of reasoning entirely analogous to that pursued in treating of the extraction of the square root, we may shew that, when the given number is expressed by more than six figures, we must point off the number into periods of three figures each, commencing at the right. Hence, for the extraction of the cube root of numbers, we have the following

RULE

I. *Separate the given number into periods of three figures each, beginning at the right hand; the left hand period will often contain less than three places of figures.*

II. *Seek the greatest perfect cube in the first period, on the left, and set its root on the right, after the manner of a quotient in division. Subtract the cube of this number from the first period, and to the remainder bring down the first figure of the next period, and call this number the dividend.*

III. *Take three times the square of the root just found for a divisor, and see how often it is contained in the dividend, and place the quotient for a second figure of the root. Then cube the number thus found, and if its cube be greater than the first two periods of the given number, diminish the last figure by* 1; *but if it be less, subtract it from the first two periods, and to the remainder bring down the first figure of the next period, for a new dividend.*

IV. *Take three times the square of the whole root for a new divisor, and seek how often it is contained in the new dividend; the quotient will be the third figure of the root. Cube the number thus found, and subtract the result from the first three periods of the given number, and proceed in a similar way for all the periods.*

If there is no remainder, the number is a *perfect cube*, and the root is exact: if there is a remainder, the number is an *imperfect cube*, and the root is exact to within less than 1.

EXAMPLES.

1. $\sqrt[3]{48228544}$ *Ans.* 364.

2. $\sqrt[3]{27054036008}$ *Ans.* 3002.

3. $\sqrt[3]{483249}$ *Ans.* 78, with a remainder 8697.

4. $\sqrt[3]{91632508641}$ *Ans.* 4508, with a remainder 20644129.

5. $\sqrt[3]{32977340218432}$ *Ans.* 32068.

Extraction of the N^{th} Root of Numbers.

142. The n^{th} root of a number is such a number as being taken n times as a factor will produce the given number, n being any positive whole number. When such a root can be exactly found, the given number is a perfect n^{th} power; all other numbers are imperfect n^{th} powers.

Let N denote any number whatever. If it is expressed by less than $n+1$ figures, and is a perfect n^{th} power, its n^{th} root will be expressed by a single figure, and may be found by

means of a table containing the n^{th} powers of the first ten numbers.

If the number is not a perfect n^{th} power, it will fall between two n^{th} powers in the table, and its root will fall between the n^{th} roots of these powers.

If the given number is expressed by more than n figures, its root will consist of a certain number of tens and a certain number of units. If we designate the tens of the root by a, and the units by b, we shall have, by the binomial formula,

$$N = (a + b)^n = a^n + na^{n-1}b + n\frac{n-1}{2}a^{n-2}b^2 +, \text{ \&c.};$$

that is, the proposed number is equal to *the n^{th} power of the tens, plus n times the product of the $n - 1^{th}$ power of the tens by the units*, plus other parts which it is not necessary to consider.

Now, as the n^{th} power of the tens, cannot be less than 1 followed by n ciphers, the last n figures on the right, cannot make a part of it. They must then be pointed off, and the n^{th} root of the greatest n^{th} power in the number on the left will be the number of tens of the required root.

Subtract the n^{th} power of the number of tens from the number on the left, and to the remainder bring down one figure of the next period on the right. If we consider the number thus found as a dividend, and take n times the $(n-1)^{th}$ power of the number of tens, as a divisor, the quotient will evidently be the number of units, or a greater number.

If the part on the left should contain more than n figures, the n figures on the right of it, must be separated from the rest, and the root of the greatest n^{th} power contained in the part on the left extracted, and so on. Hence the following

RULE.

I. *Separate the number N into periods of n figures each, beginning at the right hand; extract the n^{th} root of the greatest perfect n^{th} power contained in the left hand period, it will be the first figure of the root.*

II. *Subtract this n^{th} power from the left hand period and bring down to the right of the remainder the first figure of the next period, and call this the dividend.*

III. *Form the $n-1$ power of the first figure of the root, multiply it by n, and see how often the product is contained in the dividend: the quotient will be the second figure of the root, or something greater.*

IV. *Raise the number thus formed to the n^{th} power, then subtract this result from the two left-hand periods, and to the new remainder bring down the first figure of the next period: then divide the number thus formed by n times the $n-1$ power of the two figures of the root already found, and continue this operation until all the periods are brought down.*

EXAMPLES.

1. What is the fourth root of 531441?

$$\begin{array}{rl|l} & \dot{5}3\,144\dot{1} & 27 \\ 2^4 = & 16 & \\ \hline 4 \times 2^3 = 32 \mid & 371 & \\ (27)^4 = & 531441. & \end{array}$$

We first point off, from the right hand, the period of four figures, and then find the greatest fourth root contained in 53, the first period to the left, which is 2. We next subtract the 4th power of 2, which is 16, from 53, and to the remainder 37 we bring down the first figure of the next period. We then divide 371 by 4 times the cube of 2, which gives 11 for a quotient: but this we know is too large. By trying the numbers 9 and 8, we find them also too large: then trying 7, we find the exact root to be 27.

143. When the index of the root to be extracted is a multiple of two or more numbers, as 4, 6, . . . &c., *the root can be obtained by extracting roots of more simple degrees, successively.* To explain this, we will remark that,

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{3\times 4} = a^{12},$$

and, in general, from the definition of an exponent

$$(a^m)^n = a^m \times a^m \times a^m \times a^m \ldots = a^{m \times n}:$$

hence, *the n^{th} power of the m^{th} power of a number is equal to the mn^{th} power of this number.*

Let us see if the converse of this is also true.

Let $$\sqrt[n]{\sqrt[m]{a}} = b;$$

then raising both members to the n^{th} power, we have, from the definition of the n^{th} root,

$$\sqrt[m]{a} = b^n;$$

and by raising both members of the last equation to the m^{th} power

$$a = b^{mn}.$$

Extracting the mn^{th} root of both members of the last equation,

we have, $$\sqrt[mn]{a} = b;$$

and hence, $$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a},$$

since each is equal to b. Therefore, *the n^{th} root of the m^{th} root of any number, is equal to the mn^{th} root of that number.* And in a similar manner, it might be proved that

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

By this method we find that

1. $\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4.$

2. $\sqrt[6]{2985984} = \sqrt[3]{\sqrt{2985984}} = \sqrt[3]{1728} = 12.$

3. $\sqrt[6]{1771561} = \sqrt[3]{\sqrt{1771561}} = 11.$

4. $\sqrt[8]{1679616} = \sqrt[4]{1296} = \sqrt{\sqrt{1296}} = 6.$

REMARK.—Although the successive roots may be extracted in any order whatever, it is better to extract the roots of the lowest degree first, for then the extraction of the roots of the higher degrees, which is a more complicated operation, is effected upon numbers containing fewer figures than the proposed number.

Extraction of Roots by Approximation.

144. When it is required to extract the n^{th} root of a number which is not a *perfect n^{th} power*, the method already explained, will give only the entire part of the root, or the root to within less than 1. As to the part which is to be added, in order to com plete the root, it cannot be obtained exactly, but we can approximate to it as near as we please.

Let it be required to extract the n^{th} root of a whole number, denoted by a, to within less than a fraction $\frac{1}{p}$; that is, so near, that the error shall be less than $\frac{1}{p}$.

We observe, that we can write

$$a = \frac{ap^n}{p^n}.$$

If we denote by r the root of the greatest perfect n^{th} power in ap^n, the number $\frac{a \times p^n}{p^n} = a$, will be comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$; therefore, the $\sqrt[n]{a}$ will be comprised between the two numbers $\frac{r}{p}$ and $\frac{r+1}{p}$; and consequently, their difference $\frac{1}{p}$ will be greater than the difference between $\frac{r}{p}$ and the true root. Hence, $\frac{r}{p}$ is the required root to within less than the fraction $\frac{1}{p}$: hence,

To extract the n^{th} root of a whole number to within less than a fraction $\frac{1}{p}$, multiply the number by p^n; extract the n^{th} root of the product to within less than 1, *and divide the result by p.*

Extraction of the n^{th} Root of Fractions.

145. Since the n^{th} power of a fraction is formed by raising both terms of the fraction to the n^{th} power, we can evidently find the n^{th} root of a fraction by extracting the n^{th} root of both terms.

If both terms are not perfect n^{th} powers, the exact n^{th} root cannot be found, but we may find its approximate root to within less than the *fractional unit*, as follows:—

Let $\frac{a}{b}$ represent the given fraction. If we multiply both terms by

$$b^{n-1}, \quad \text{it becomes,} \quad \frac{a}{b} = \frac{ab^{n-1}}{b^n}.$$

Let r denote the n^{th} root of the greatest n^{th} power in ab^{n-1}; then $\frac{ab^{n-1}}{b^n}$ will be comprised between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$; and consequently, $\frac{r}{b}$ will be the n^{th} root of $\frac{a}{b}$ to within less than the fraction $\frac{1}{b}$; therefore,

Multiply the numerator by the $(n-1)^{th}$ power of the denominator and extract the n^{th} root of the product: Divide this root by the denominator of the given fraction, and the quotient will be the approximate root.

When a greater degree of exactness is required than that indicated by $\frac{1}{b}$, extract the n^{th} root of ab^{n-1} to within any fraction $\frac{1}{p}$; and designate this root by $\frac{r'}{p}$. Now, since $\frac{r'}{p}$ is the root of the numerator to within less than $\frac{1}{p}$, it follows, that $\frac{r'}{bp}$ is the true root of the fraction to within less than $\frac{1}{bp}$.

EXAMPLES.

1. Suppose it were required to extract the cube root of 15 to within less than $\frac{1}{12}$. We have

$$15 \times 12^3 = 15 \times 1728 = 25920.$$

Now, the cube root of 25920, to within less than 1 is 29: hence, the required root is,

$$\frac{29}{12} = 2\frac{5}{12}.$$

2. Extract the cube root of 47, to within less than $\frac{1}{20}$.

We have,

$$47 \times 20^3 = 47 \times 8000 = 376000.$$

Now, the cube root of 376000, to within less than 1, is 72; hence, $\sqrt[3]{47} = \frac{72}{20} = 3\frac{12}{20}$, to within less than $\frac{1}{20}$.

3. Find the value of $\sqrt[3]{25}$, to within less than .001.

To do this, multiply 25 by the cube of 1000, or 1000000000, which gives 25000000000. Now, the cube root of this number. is 2920; hence,

$$\sqrt[3]{25} = 2.920 \text{ to within less than } .001.$$

Hence, to extract the cube root of a whole number to within less than a given decimal fraction, we have the following

RULE.

Annex three times as many ciphers to the number, as there are decimal places in the required root; extract the cube root of the number thus formed to within less than 1, *and point off from the right of this root the required number of decimal places.*

146. We will now explain the method of *extracting the cube root of a decimal fraction.*

Suppose it is required to extract the cube root of 3.1415.

Since the denominator, 10000, of this fraction, is not a perfect cube, make it one, by multiplying it by 100; *this is equivalent to annexing two ciphers to the proposed decimal,* which then becomes, 3.141500. Extract the cube root of 3141500, that is, of the number considered independent of the decimal point to within less than 1; this gives 146. Then dividing by 100, or $\sqrt[3]{1000000}$, and we find,

$$\sqrt[3]{3.1415} = 1.46 \text{ to within less than } 0.01.$$

Hence, to extract the cube root of a decimal fraction, we have the following

RULE.

Annex ciphers till the whole number of decimal places is equal to three times the number of required decimal places in the root. Then extract the root as in whole numbers, and point off the required number of decimal places.

To extract the cube root of a vulgar fraction to within less than a given decimal fraction, the most simple method is,

To reduce the proposed fraction to a decimal fraction, continuing the division until the number of decimal places is equal to three times the number required in the root.

The question is then reduced to extracting the cube root of a decimal fraction.

Suppose it is required to find the sixth root of 23, to within less than 0.01.

Applying the rule of Art. 144 to this example, we multiply 23 by $(100)^6$, or annex *twelve* ciphers to 23; then extract the sixth root of the number thus formed to within less than 1, and divide this root by 100, or point off two decimal places on the right: we thus find,

$$\sqrt[6]{23} = 1.68, \text{ to within less than } 0.01.$$

EXAMPLES.

1. Find the $\sqrt[3]{473}$ to within less than $\frac{1}{20}$. *Ans.* $7\frac{3}{4}$.

2. Find the $\sqrt[3]{79}$ to within less than .0001. *Ans.* 4.2908.

3. Find the $\sqrt[6]{13}$ to within less than .01. *Ans.* 1.53.

4. Find the $\sqrt[3]{3.00415}$ to within less than .0001. *Ans.* 1.4429.

5. Find the $\sqrt[3]{0.00101}$ to within less than .01. *Ans.* 0.10.

6. Find the $\sqrt[3]{\frac{14}{25}}$ to within less than .001. *Ans.* 0.824.

Extraction of Roots of Algebraic Quantities.

147. Let us first consider the case of monomials, and in order to deduce a rule for extracting the n^{th} root, let us examine the law for the formation of the n^{th} power.

From the definition of a power, it follows that each factor of the root will enter the power, as many times as there are units in the exponent of the power. That is, to form the n^{th} power of a monomial,

We form the n^{th} power of the co-efficient for a new co-efficient, and write after this, each letter affected with an exponent equal to n times its primitive exponent.

Conversely, we have for the extraction of the n^{th} root of a monomial, the following

RULE.

Extract the n^{th} root of the numerical co-efficient for a new co-efficient, and after this write each letter affected with an exponent equal to $\frac{1}{n}$th of its exponent in the given monomial; the result will be the required root.

Thus, $\sqrt[3]{64a^9b^3c^6} = 4a^3bc^2$; and $\sqrt[4]{16a^8b^{12}c^4} = 2a^2b^3c$.

From this rule we perceive, that in order that a monomial may be a perfect n^{th} power:

1st. Its co-efficient must be a perfect n^{th} power; and

2d. The exponent of each letter must be divisible by n.

It will be shown, hereafter, how the expression for the root of a quantity, which is not a perfect power, is reduced to its simplest form.

148. Hitherto, in finding the power of a monomial, we have paid no attention to the sign with which the monomial may be affected. It has already been shown, that whatever be the sign of a monomial, *its square is always positive.*

Let n be any whole number; then every power of an even degree, as $2n$, can be considered as the n^{th} power of the square; that is, $(a^2)^n = a^{2n}$: hence, it follows,

That every power of an even degree, will be essentially positive, whether the quantity itself be positive or negative.

Thus, $$(\pm 2a^2b^3c)^4 = +16a^8b^{12}c^4.$$

Again, as every power of an uneven degree, $2n+1$, is but the product of the power of an even degree, $2n$, by the first power; it follows that,

Every power of a monomial, of an uneven degree, has the same sign as the monomial itself.

Hence, $(+4a^2b)^3 = +64a^6b^3$; and $(-4a^2b)^3 = -64a^6b^3$.

From the preceding reasoning, we conclude,

1st. *That when the index of the root of a monomial is uneven, the root will be affected with the same sign as the monomial.*

Thus,

$$\sqrt[3]{+8a^3} = +2a; \quad \sqrt[3]{-8a^3} = -2a; \quad \sqrt[5]{-32a^{10}b^5} = -2a^2b.$$

2d. *When the index of the root is even, and the monomial a positive quantity, the root has both the signs + and −.*

Thus, $\sqrt[4]{81a^4b^{12}} = \pm 3ab^3$; $\sqrt[6]{64a^{18}} = \pm 2a^3$.

3d. *When the index of the root is even, and the monomial negative, the root is impossible;*

For, there is no quantity which, being raised to a power of an even degree, will give a negative result. Therefore,

$$\sqrt[4]{-a}, \quad \sqrt[6]{-b}, \quad \sqrt[8]{-c},$$

are symbols of operations which it is impossible to execute They are *imaginary expressions.*

EXAMPLES.

1. What is the cube root of $8a^6b^3c^{12}$? *Ans.* $2a^2bc^4$.
2. What is the 4th root of $81a^4b^8c^{16}$? *Ans.* $3ab^2c^4$.
3. What is the 5th root of $-32a^5c^{10}d^{15}$? *Ans.* $-2ac^2d^3$.
4. What is the cube root of $-125a^9b^6c^3$? *Ans.* $-5a^3b^2c$.

Extraction of the n^{th} Root of Polynomials.

148. Let N denote any polynomial whatever, arranged with reference to a certain letter. Now, the n^{th} power of a polynomial is the continued product arising from taking the polynomial n times as a factor: hence, the *first term* of the product, when arranged with reference to a certain letter, is the n^{th} power of the first term of the polynomial, arranged with reference to the same letter.

Therefore, the n^{th} root of the first term of such a product, will be the first term of the n^{th} root of the product.

Let us denote the first term of the n^{th} root of N by r, and the following terms, arranged with reference to the leading letter of the polynomial, by r', r'', r''', &c. We shall have,

$$N = (r + r' + r'' + \ldots \text{ \&c.})^n;$$

or, if we designate the sum of all the terms after the first by s,

$$N = (r + s)^n = r^n + nr^{n-1}s + \text{ \&c.},$$
$$= r^n + nr^{n-1}(r' + r'' + \text{\&c.}) + \text{\&c.}$$

If now, we subtract r^n from N, and designate the remainder by R, we shall have,

$$R = N - r^n = nr^{n-1}r' + nr^{n-1}r'' + \text{\&c.},$$

which remainder will evidently be arranged with reference to the leading letter of the polynomial; therefore, the first term will contain a higher power of that letter than either of the succeeding terms, and cannot be reduced with any of them. Hence, if we divide the first term of the first remainder, by n times the $(n-1)^{th}$ power of the first term of the root, the quotient will be the second term of the root.

If now, we place $r + r' = u$, and denote the sum of the succeeding terms of the root by s', we shall have,

$$N = (u + s')^n = u^n + nu^{n-1}s' + \text{\&c}$$

If now, we subtract u^n from N, and denote the remainder by R', we shall have,

$$R' = N - u^n = n(r + r')^{n-1}s' + \&c.,$$
$$= nr^{n-1}(r'' + r''' + \&c.) + \&c.,$$
$$= nr^{n-1}r'' + \&c.$$

If we divide the first term of this remainder by n times the $(n-1)^{th}$ power of the first term of the root, we shall have the third term of the root. If we continue the operation, we shall find that the first term of any new remainder, divided by n times the $(n-1)^{th}$ power of the first term of the root, will give a new term of the root.

It may be remarked, that since the first term of the first remainder is the same as the second term of the given polynomial, we can find the second term of the root, by dividing the second term of the given polynomial by n times the $(n-1)^{th}$ power of the first term.

Hence, for the extraction of the n^{th} root of a polynomial, we have the following

RULE.

I. *Arrange the given polynomial with reference to one of its letters, and extract the n^{th} root of the first term; this will be the first term of the root.*

II. *Divide the second term by n times the $(n-1)^{th}$ power of the first term of the root; the quotient will be the second term of the root.*

III. *Subtract the n^{th} power of the sum of the two terms already found from the given polynomial, and divide the first term of the remainder by n times the $(n-1)^{th}$ power of the first term of the root; the quotient will be the third term of the root.*

IV. *Continue this operation till a remainder is found equal to 0, or, till one is found whose first term is not divisible by n times the $(n-1)^{th}$ power of the first term of the root: in the former case the root is exact, and the given polynomial a perfect n^{th} power; in the latter case, the polynomial is an imperfect n^{th} power.*

149. Let us apply the foregoing rule to the following

EXAMPLES.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{lll} & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & \underline{|\, x^2-2x+1} \\ (x^2-2x)^3= & \underline{x^6-6x^5+12x^4-\ 8x^3} & \quad 3x^4 \\ \text{1st rem.} & \qquad 3x^4-12x^3+\ \ \&c. & \\ (x^2-2x+1)^3= & x^6-6x^5+15x^4-20x^3+15x^2-6x+1. & \end{array}$$

In this example, we first extract the cube root of x^6, which gives x^2, for the first term of the root. Squaring x^2, and multiplying by 3, we obtain the divisor $3x^4$: this is contained in the second term $-6x^5$, $-2x$ times. Then cubing the part of the root found, and subtracting, we find that the first term of the remainder $3x^4$, contains the divisor once. Cubing the whole root found, we find the cube equal to the given polynomial. Hence, x^2-2x+1, is the exact cube root.

2. Find the cube root of

$$x^6+6x^5-40x^3+96x-64.$$

3. Find the cube root of

$$8x^6-12x^5+30x^4-25x^3+30x^2-12x+8.$$

4. Find the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$

$$\begin{array}{ll|l} & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4 & \underline{\ 2a-3x\ } \\ (2a-3x)^4= & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4 & 4\times(2a)^3=32a^3. \end{array}$$

We first extract the 4th root of $16a^4$, which is $2a$. We then raise $2a$ to the third power, and multiply by 4, the index of the root; this gives the divisor $32a^3$. This divisor is contained in the second term $-96a^3x$, $-3x$ times, which is the second term of the root. Raising the whole root found to the 4th power we find the power equal to the given polynomial.

5. What is the 4th root of the polynomial,

$$81a^8c^4+16b^4d^4-96a^2cb^3d^3-216a^6c^3bd+216a^4c^2b^2d^2.$$

6. Find the 5th root of

$$32x^5-80x^4+80x^3-40x^2+10x-1.$$

Find $32x^5$

Transformation of Radicals of any Degree.

150. The principles demonstrated in Art. 104, are general. For, let $\sqrt[n]{a}$ and $\sqrt[n]{b}$, be any two radicals of the n^{th} degree, and denote their product by p. We shall have,

$$\sqrt[n]{a} \times \sqrt[n]{b} = p \quad - \quad - \quad - \quad (1).$$

By raising both members of this equation to the n^{th} power, we find

$$(\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = p^n, \quad \text{or} \quad ab = p^n;$$

whence, by extracting the n^{th} root of both members,

$$\sqrt[n]{ab} = p \quad - \quad - \quad - \quad (2).$$

Since the second members of equations (1) and (2) are the same, their first members are equal, whence,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}: \quad \text{hence,}$$

1st. *The product of the n^{th} roots of two quantities, is equal to the n^{th} root of the product of the quantities.*

Denote the quotient of the given radicals by q, we shall have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = q \quad - \quad - \quad - \quad - \quad (1);$$

and by raising both members to the n^{th} power,

$$\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = q^n \quad \text{or} \quad \frac{a}{b} = q^n;$$

whence, by extracting the n^{th} root of the two members, we have,

$$\sqrt[n]{\frac{a}{b}} = q \quad - \quad - \quad - \quad - \quad (2).$$

The second members of equations (1) and (2) being the same, their first members are equal, giving

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}; \quad \text{hence,}$$

2d. *The quotient of the n^{th} roots of two quantities, is equal to the n^{th} root of the quotient of the quantities.*

151. Let us apply the first principle of article 150, to the simplification of the radicals in the following

EXAMPLES.

1. Take the radical $\sqrt[3]{54a^4b^3c^2}$. This may be written,

$$\sqrt[3]{54a^4b^3c^2} = \sqrt[3]{27a^3b^3} \times \sqrt[3]{2ac^2} = 3ab\sqrt[3]{2ac^2}.$$

2. In like manner,

$$\sqrt[3]{8a^2} = 2\sqrt[3]{a^2}; \quad \text{and} \quad \sqrt[4]{48a^5b^8c^6} = 2ab^2c\sqrt[4]{3ac^2};$$

3. Also,

$$\sqrt[6]{192a^7bc^{12}} = \sqrt[6]{64a^6c^{12}} \times \sqrt[6]{3ab} = 2ac^2\sqrt[6]{3ab}.$$

In the expressions, $3ab\sqrt[3]{2ac^2}$, $2\sqrt[3]{a^2}$, $2ab^2c\sqrt[4]{3ac^2}$, each quantity placed before the radical, is called a *co-efficient* of the radical.

Since we may simplify any radical in a similar manner, we have, for the simplification of a radical of the n^{th} degree, the following

RULE.

Resolve the quantity under the radical sign into two factors, one of which shall be the greatest perfect n^{th} power which enters it; extract the n^{th} root of this factor, and write the root without the radical sign, under which, leave the other factor.

Conversely, *a co-efficient may be introduced under the radical sign, by simply raising it to the n^{th} power, and writing it as a factor under the radical sign.*

Thus, $3ab\sqrt[3]{2ac^2} = \sqrt[3]{27a^3b^3} \times \sqrt[3]{2ac^2} = \sqrt[3]{54a^4b^3c^2}.$

152. By the aid of the principles demonstrated in article **143**, we are enabled to make another kind of simplification.

Take, for example, the radical $\sqrt[6]{4a^2}$; from the principles referred to, we have,

$$\sqrt[6]{4a^2} = \sqrt[3]{\sqrt{4a^2}},$$

and as the quantity under the radical sign of the second degree is a perfect square, its root can be extracted: hence,

$$\sqrt[6]{4a^2} = \sqrt[3]{2a}.$$

In like manner,

$$\sqrt[4]{36a^2b^2} = \sqrt{\sqrt{36a^2b^2}} = \sqrt{6ab}.$$

In general,

$$\sqrt[mn]{a^n} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[m]{a};$$

that is, when the index of a radical is a multiple of any number n, and the quantity under the radical sign is an exact n^{th} power, *We can, without changing the value of the radical, divide its index by n, and extract the n^{th} root of the quantity under the sign.*

153. Conversely, *The index of a radical may be multiplied by any number, provided we raise the quantity under the sign to a power of which this number is the exponent.*

For, since a is the same thing as $\sqrt[n]{a^n}$, we have,

$$\sqrt[m]{a} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[mn]{a^n}.$$

154. The last principles enable us to reduce two or more radicals of different degrees, to equivalent radicals having a common index.

For example, let it be required to reduce the two radicals

$$\sqrt[3]{2a} \quad \text{and} \quad \sqrt[4]{(a+b)}$$

to the same index.

By multiplying the index of the first by 4, the index of the second, and raising the quantity $2a$ to the fourth power; then multiplying the index of the second by 3, the index of the first, and cubing $a+b$, the value of neither radical will be changed, and the expressions will become

$$\sqrt[3]{2a} = \sqrt[12]{2^4a^4} = \sqrt[12]{16a^4}; \quad \text{and} \quad \sqrt[4]{(a+b)} = \sqrt[12]{(a+b)^3},$$

and similarly for other radicals: hence, to reduce radicals to a common index, we have the following

RULE.

Multiply the index of each radical by the product of the indices of all the other radicals, and raise the quantity under each radical sign to a power denoted by this product.

This rule, which is analogous to that given for the reduction of fractions to a common denominator, is susceptible of similar modifications.

For example, reduce the radicals

$$\sqrt[4]{a}, \qquad \sqrt[6]{5b}, \qquad \sqrt[8]{a^2+b^2},$$

to a common index.

Since 24 is the least common multiple of the indices, 4, 6, and 8, it is only necessary to multiply the first by 6, the second by 4, and the third by 3, and to raise the quantities under each radical sign to the 6th, 4th, and 3d powers, respectively, which gives

$$\sqrt[4]{a} = \sqrt[24]{a^6}; \quad \sqrt[6]{5b} = \sqrt[24]{5^4b^4}, \quad \sqrt[8]{a^2+b^2} = \sqrt[24]{(a^2+b^2)^3}.$$

Addition and Subtraction of Radicals of any Degree.

155. We first reduce the radicals to their simplest form by the aid of the preceding rules, and then if they are *similar*, in order to add them together, we *add their co-efficients, and after this sum write the common radical;* if they are not similar, the addition can only be indicated.

Thus, $\qquad 3\sqrt[3]{b} + 2\sqrt[3]{b} = 5\sqrt[3]{b}.$

EXAMPLES.

1. Find the sum of $\sqrt{48ab^2}$ and $b\sqrt{75a}$. *Ans.* $9b\sqrt{3a}$.
2. Find the sum of $3\sqrt[6]{4a^2}$ and $2\sqrt[3]{2a}$. *Ans.* $5\sqrt[3]{2a}$.
3. Find the sum of $2\sqrt{45}$ and $3\sqrt{5}$. *Ans.* $9\sqrt{5}$.

155*. In order to subtract one radical from another when they are similar,

Subtract the co-efficient of the subtrahend from the co-efficient of the minuend, and write this difference before the common radical.

Thus, $3a\sqrt[4]{b} - 2c\sqrt[4]{b} = (3a - 2c)\sqrt[4]{b}$;

but, $2ab\sqrt{cd} - 5ab\sqrt{c}$ are irreducible.

1. From $\sqrt[3]{8a^3b + 16a^4}$ subtract $\sqrt[3]{b^4 + 2ab^3}$.

Ans. $(2a - b)\sqrt[3]{b + 2a}$.

2. From $3\sqrt[6]{4a^2}$ subtract $2\sqrt[3]{2a}$. *Ans.* $\sqrt[3]{2a}$.

Multiplication of Radicals of any Degree.

156. We have shown that all radicals may be reduced to equivalent ones having a common index; we therefore suppose this transformation made.

Now, let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ denote any two radicals of the same degree. Their product may be denoted thus,

$$a\sqrt[n]{b} \times c\sqrt[n]{d};$$

or since the order of the factors may be changed without affecting the value of the product, we may write it,

$ac \times \sqrt[n]{b} \times \sqrt[n]{d}$ or (Art. 150), since $\sqrt[n]{d} \times \sqrt[n]{b} = \sqrt[n]{db}$;

we have finally,

$$a\sqrt[n]{b} \times c\sqrt[n]{d} = ac\sqrt[n]{bd};$$

hence, for the multiplication of radicals of any degree, we have the following

RULE.

I. *Reduce the radicals to equivalent ones having a common index.*

II *Multiply the co-efficients together for a new co-efficient; after this write the radical sign with the common index, placing under it the product of the quantities under the radical signs in the two factors; the result is the product required.*

EXAMPLES.

1. The product

$$2a\sqrt[3]{\frac{a^2+b^2}{c}} \times -3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}} = -6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}$$

$$= -\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}}.$$

2. The product

$$3a\sqrt[4]{8a^2} \times 2b\sqrt[4]{4a^2c} = 6ab\sqrt[4]{32a^4c} = 12a^2b\sqrt[4]{2c}.$$

3. The product

$$\frac{3}{4}\sqrt[3]{\frac{1}{3}} \times \frac{1}{4}\sqrt[3]{\frac{1}{7}} = \frac{3}{16}\sqrt[3]{\frac{1}{21}}.$$

4. The product

$$3a\sqrt[6]{b} \times 5b\sqrt[8]{2c} = 15ab \times \sqrt[24]{8b^4c^3}.$$

5. Multiply $\sqrt{2} \times \sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}}$.

Ans. $\sqrt[12]{8}$.

6. Multiply $2\sqrt{15}$ by $3\sqrt[3]{10}$.

Ans. $6\sqrt[6]{337500}$.

7. Multiply $4\sqrt[5]{\frac{2}{3}}$ by $2\sqrt{\frac{3}{4}}$.

Ans. $8\sqrt[10]{\frac{27}{256}}$.

8. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$, together.

Ans. $\sqrt[12]{648000}$.

9. Multiply $\sqrt[7]{\frac{4}{3}}$, $\sqrt[3]{\frac{1}{2}}$ and $\sqrt[14]{6}$, together.

Ans. $\sqrt[42]{\frac{2}{27}}$.

10. Multiply $\left(4\sqrt{\frac{7}{3}} + 5\sqrt{\frac{1}{2}}\right)$ by $\left(\sqrt{\frac{7}{3}} + 2\sqrt{\frac{1}{2}}\right)$.

Ans. $\frac{43}{3} + \frac{13}{6}\sqrt{42}$.

Division of Radicals of any Degree.

157. We will suppose, as in the last article, that the radicals have been reduced to equivalent ones having a common index.

Let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ represent any two radicals of the nth degree. The quotient of the first by the second may be written,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}} = \frac{a}{c} \times \frac{\sqrt[n]{b}}{\sqrt[n]{d}},$$

or, since $\frac{\sqrt[n]{b}}{\sqrt[n]{d}} = \sqrt[n]{\frac{b}{d}}$ (Art. 150), we have,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}} = \frac{a}{c} \times \sqrt[n]{\frac{b}{d}}.$$

Hence, to divide one radical by another, we have the following

RULE.

I. *Reduce the radicals to equivalent ones having a common index.*

II. *Divide the co-efficient of the dividend by that of the divisor for a new co-efficient; after this write the radical sign with the common index, and place under it the quotient obtained by dividing the quantity under the radical sign in the dividend by that in the divisor; the result will be the quotient required.*

EXAMPLES.

1. What is the quotient of $c\sqrt[3]{a^2b^2+b^4}$ divided by $d\sqrt[3]{\frac{a^2-b^2}{8b}}$?

$$\frac{c}{d} \times \frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\frac{a^2-b^2}{8b}}} = \frac{c}{d}\sqrt[3]{\frac{8b\,(a^2b^2+b^4)}{a^2-b^2}} = \frac{2cb}{d}\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$$

2. Divide $2\sqrt{3} \times \sqrt[3]{4}$ by $\frac{1}{2}\sqrt[4]{2} \times \sqrt[3]{3}$.

Ans. $4\sqrt[12]{288}$.

3. Divide $\sqrt{\sqrt{\tfrac{1}{2}} \times 2\sqrt[3]{3}}$ by $\sqrt{4\sqrt[3]{2} \times \sqrt{3}}$.

Ans. $\frac{1}{2}\sqrt[12]{\frac{2}{3}}$.

4. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\left(\sqrt{2}+3\sqrt{\frac{1}{2}}\right)$.

Ans. $\frac{1}{10}$.

5. Divide 1 by $\sqrt[4]{a}+\sqrt[4]{b}$.

Ans. $\frac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}$.

6. Divide $\sqrt[4]{a}+\sqrt[4]{b}$ by $\sqrt[4]{a}-\sqrt[4]{b}$.

$$Ans.\quad \frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}.$$

Formation of Powers of Radicals of any Degree.

158. Let $a\sqrt[n]{b}$ represent any radical of the n^{th} degree. Then we may raise this radical to the m^{th} power, by taking it m times as a factor; thus,

$$a\sqrt[n]{b}\times a\sqrt[n]{b}\ \text{- - - - - - - -}\ a\sqrt[n]{b}.$$

But, by the rule for multiplication, this continued product is equal to $a^m\sqrt[n]{b^m}$; whence,

$$(a\sqrt[n]{b})^m = a^m\sqrt[n]{b^m} \quad \text{- - - - -} \quad (1).$$

We have then, to raise a radical to any power, the following

RULE.

Raise the co-efficient to the required power for a new co-efficient; after this write the radical sign with its primitive index, placing under it the required power of the quantity under the radical sign in the given expression; the result will be the power required.

EXAMPLES.

1. $(\sqrt[4]{4a^3})^2 = \sqrt[4]{(4a^3)^2} = \sqrt[4]{16a^6} = 2a\sqrt[4]{a^2} = 2a\sqrt{a}.$

2. $(3\sqrt[3]{2a})^5 = 3^5\sqrt[3]{(2a)^5} = 243\sqrt[3]{32a^5} = 486a\sqrt[3]{4a^2}.$

When the index of the radical is a multiple of the exponent of the power to which it is to be raised, the result can be simplified.

For, $\sqrt[4]{2a}=\sqrt{\sqrt{2a}}$ (Art. 152): hence, in order to square $\sqrt[4]{2a}$, we have only to omit the first radical sign, which gives

$$(\sqrt[4]{2a})^2 = \sqrt{2a}.$$

Again, to square $\sqrt[6]{3b}$, we have $\sqrt[6]{3b}=\sqrt{\sqrt[3]{3b}}$: hence,

$$(\sqrt[6]{3b})^2 = \sqrt[3]{3b};\ \text{hence,}$$

When the index of the radical is divisible by the exponent of the power to which it is to be raised, perform the division, leaving the quantity under the radical sign unchanged.

Extraction of Roots of Radicals of any Degree.

159. By extracting the m^{th} root of both members of equation (1), of the preceding article, we find,

$$\sqrt[m]{a^m \times \sqrt[n]{b^m}} = a\sqrt[n]{b};$$

Whence we see, that to extract any root of a radical of any degree, we have the following

RULE.

Extract the required root of the co-efficient for a new co-efficient; after this write the radical sign with its primitive index, under which place the required root of the quantity under the radical sign in the given expression; the result will be the root required.

EXAMPLES.

1. Find the cube root of $8\sqrt[4]{27}$. *Ans.* $2\sqrt[4]{3}$.

2. Find the fourth root of $\frac{1}{16}\sqrt[3]{256}$. *Ans.* $\frac{1}{2}\sqrt[3]{4}$.

159*. If, however, the required root of the quantity under the radical sign cannot be exactly found, we may proceed in the following manner. If it be required to find the m^{th} root of $c\sqrt[n]{d}$, the operation may be indicated thus,

$$\sqrt[m]{c\sqrt[n]{d}} = \sqrt[m]{c}\sqrt[m]{\sqrt[n]{d}};$$

but $\sqrt[m]{\sqrt[n]{d}} = \sqrt[mn]{d}$, whence, by substituting in the previous equation,

$$\sqrt[m]{c\sqrt[n]{d}} = \sqrt[m]{c}\sqrt[mn]{d}:$$

Consequently, when we cannot extract the required root of the quantity under the radical sign,

Extract the required root of the co-efficient for a new co-efficient; after this, write the radical sign, with an index equal to the product of its primitive index by the index of the required root, leaving the quantity under the radical sign unchanged.

EXAMPLES.

1. $\sqrt[3]{\sqrt[4]{3c}} = \sqrt[12]{3c}$; and, $\sqrt{\sqrt[3]{5c}} = \sqrt[6]{5c}$.

When the quantity under the radical is a perfect power, of the *degree of either of the roots to be extracted*, the result can be simplified.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}} = \sqrt[4]{\sqrt[3]{8a^3}} = \sqrt[4]{2a}$.

In like manner, $\sqrt{\sqrt[5]{9a^2}} = \sqrt[5]{\sqrt{9a^2}} = \sqrt[5]{3a}$.

2. Find the cube root of $\frac{1}{27}\sqrt{3}$. *Ans.* $\frac{1}{3}\sqrt[6]{3}$.

3. Find the cube root of $\frac{1}{8}\sqrt{2ab^2}$. *Ans.* $\frac{1}{2}\sqrt[6]{2ab^2}$.

Different Roots of the same Power.

160. The rules just demonstrated depend upon the principle, that *if two quantities are equal, the like roots of those quantities are also equal.*

This principle is true so long as we regard the term *root* in its general sense, but when the term is used in a restricted sense, it requires some modification. This modification is particularly necessary in operating upon imaginary expressions, which are not roots, strictly speaking, but mere indications of operations which it is impossible to perform. Before pointing out these modifications, it will be shown, that every quantity has more than one *cube root*, *fourth root*, &c.

It has already been shown, that every quantity has two square roots, equal, with contrary signs.

1. Let x denote the general expression for the cube root of a, and let p denote the numerical value of this root; we have the equations

$$x^3 = a, \quad \text{and} \quad x^3 = p^3.$$

The last equation is satisfied by making $x = p$.

Observing that the equation $x^3 = p^3$ can be put under the form $x^3 - p^3 = 0$, and that the expression $x^3 - p^3$ is divisible by $x - p$, giving the quotient, $x^2 + px + p^2$, the above equation can be placed under the form

$$(x - p)(x^2 + px + p^2) = 0.$$

Now, every value of x that will satisfy this equation, will satisfy the first equation. But this equation can be satisfied by supposing

$$x - p = 0, \quad \text{whence,} \quad x = p;$$

or by supposing

$$x^2 + px + p^2 = 0,$$

from which we have,

$$x = -\frac{p}{2} \pm \frac{p}{2}\sqrt{-3}, \quad \text{or} \quad x = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right);$$

hence, we see, that *there are three different algebraic expressions for the cube root of* a, *viz:*

$$p, \quad p\left(\frac{-1 + \sqrt{-3}}{2}\right), \quad \text{and} \quad p\left(\frac{-1 - \sqrt{-3}}{2}\right).$$

2. Again, solve the equation

$$x^4 = p^4,$$

in which p denotes the arithmetical value of $\sqrt[4]{a}$.

This equation can be put under the form

$$x^4 - p^4 = 0;$$

which reduces to

$$(x^2 - p^2)(x^2 + p^2) = 0;$$

and this equation can be satisfied, by supposing

$$x^2 - p^2 = 0; \quad \text{whence,} \quad x = \pm p;$$

or by supposing

$$x^2 + p^2 = 0, \quad \text{whence,} \quad x = \pm\sqrt{-p^2} = \pm p\sqrt{-1}.$$

We therefore obtain four *different algebraic* expressions for the fourth root of a.

3. As another example, solve the equation

$$x^6 - p^6 = 0.$$

This equation can be put under the form

$$(x^3 - p^3)(x^3 + p^3) = 0;$$

which may be satisfied by making either of the factors equal to zero.

But, $x^3 - p^3 = 0$, gives

$$x = p, \quad \text{and} \quad x = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right).$$

And if in the equation $x^3 + p^3 = 0$, we make $p = -p'$, it becomes $x^3 - p'^3 = 0$, from which we deduce

$$x = p', \quad \text{and} \quad x = p'\left(\frac{-1 \pm \sqrt{-3}}{2}\right);$$

or, substituting for p' its value $-p$,

$$x = -p, \quad \text{and} \quad x = -p\left(\frac{-1 \pm \sqrt{-3}}{2}\right).$$

Therefore, x in the equation

$$x^6 - p^6 = 0,$$

and consequently, the 6th root of a, admits of *six different algebraic expressions.* If we make

$$a = \frac{-1 + \sqrt{-3}}{2}, \quad \text{and} \quad a' = \frac{-1 - \sqrt{-3}}{2},$$

these expressions become

$$p, \quad ap, \quad a'p, \quad -p, -ap, -a'p.$$

It may be demonstrated, generally, that there are as many different expressions for the n^{th} root of a quantity as there are units in n. If n is an even number, and the quantity is positive, two of the expressions will be real, and equal, with contrary signs; all the rest will be imaginary: if the quantity is negative, they will all be imaginary.

If n is odd, one of the expressions will be real, and all the rest will be imaginary.

161. If in the preceding article we make $a = 1$, we shall find the expressions for the *second*, *third*, *fourth*, &c., *roots* of 1.

Thus, $+1$ and -1 are the square roots of 1.

Also, $+1$, $\dfrac{-1+\sqrt{-3}}{2}$, and $\dfrac{-1-\sqrt{-3}}{2}$.

are the *cube* roots of 1:

And $+1$, -1, $+\sqrt{-1}$ and $-\sqrt{-1}$, are the *fourth* roots of 1, &c., &c.

Rules for Imaginary Expressions.

162. We shall now explain the modification of the rules for operating upon radicals when applied to imaginary expressions.

The product of $\sqrt{-a}$ by $\sqrt{-a}$, by the rule of Art. 156, would be $\sqrt{+a^2}$. Now, $\sqrt{+a^2}$ is equal to $\pm a$, whence there is an apparent uncertainty as to the sign of a. The true product, however, is $-a$, since, from the definition of the square root of a quantity, we have only to omit the radical sign, to obtain the quantity.

Again, let it be required to form the product

$$\sqrt{-a} \times \sqrt{-b}.$$

By the rule of Art. 156, we shall have

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{+ab};$$

but the true result is $-\sqrt{ab}$, so long as both the radicals $\sqrt{-a}$ and $\sqrt{-b}$ are affected with the sign $+$.

For, $\sqrt{-a} = \sqrt{a}.\sqrt{-1}$; and $\sqrt{-b} = \sqrt{b}.\sqrt{-1}$;

hence,

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{a}.\sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab}\,(\sqrt{-1})^2$$
$$= \sqrt{ab} \times -1 = -\sqrt{ab}.$$

In a similar manner, we treat all other imaginary expressions of the second degree; that is, we first reduce them to the form of $a\sqrt{-1}$, in which the co-efficient of $\sqrt{-1}$ is real, and then proceed as indicated in the last article.

162*. For convenience, in the application of the preceding principle, we deduce the different powers of $\sqrt{-1}$, as follows:

$$(\sqrt{-1})^1 = \sqrt{-1},$$

$$(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1,$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -\sqrt{-1},$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = +1.$$

The fifth power is evidently the same as the first power; the sixth power the same as the second; the seventh the same as the third, and so on, indefinitely.

163. If it is required to find the product of $\sqrt[4]{-a}$ and $\sqrt[4]{-b}$, we should get, by applying the rule of Art. 156.

$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{+ab}$, but this is not the true result.

For, placing the quantities under the form

$$\sqrt[4]{a} \times \sqrt[4]{-1} \quad \text{and} \quad \sqrt[4]{b} \times \sqrt[4]{-1},$$

and proceeding to form the product, we find

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{a} \times \sqrt[4]{b} \times (\sqrt[4]{-1})^2 = \sqrt[4]{ab} \times \sqrt{-1},$$

since, $(\sqrt[4]{-1})^2 = \left(\sqrt{\sqrt{-1}}\right)^2 = \sqrt{-1}$ from the definition of a root.

Hence, generally, when we have to apply the rules for radicals to imaginary expressions of the fourth degree, transform them, so that the only factor under the radical sign shall be -1, and then proceed as in the above example.

Let us illustrate this remark, by showing that $\dfrac{-1+\sqrt{-3}}{2}$ is an expression for the cube root of 1, or that, in the restricted sense, it is a cube root of 1.

We have

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^3=\left(\frac{-1+\sqrt{3}\sqrt{-1}}{2}\right)^3,$$

$$=\frac{(-1)^3+3.(-1)^2.\sqrt{3}.\sqrt{-1}+3.(-1).(\sqrt{3})^2.(\sqrt{-1})^2+(\sqrt{3})^3.(\sqrt{-1})^3}{8}$$

$$=\frac{-1+3\sqrt{3}.\sqrt{-1}-3\times-3-3\sqrt{3}.\sqrt{-1}}{8}=\frac{8}{8}=1.$$

In like manner, we may show, that $\dfrac{-1-\sqrt{-3}}{2}$ is another expression for the cube root of 1, when understood in the restricted sense. It may be remarked that either of these expressions is equal to the square of the other, as may easily be shown.

Of Fractional and Negative Exponents.

164. We have yet to explain a system of notation by means of which operations upon radical quantities may be greatly simplified.

We have seen, in order to extract the n^{th} root of the quantity a^m, that when m is a multiple of n, we have simply to divide the *exponent* of the power, by the *index* of the root to be extracted, thus,

$$\sqrt[n]{a^m}=a^{\frac{m}{n}}.$$

When m is not a multiple of n, it has been agreed to retain the notation,

$$\sqrt[n]{a^m}=a^{\frac{m}{n}},$$

these two being regarded as equivalent expressions, and both indicating the n^{th} root of the m^{th} power of a, or what is the same thing, the m^{th} power of the n^{th} root of a; and generally,

When any quantity is written with a fractional exponent, the numerator of the fraction denotes the power to which the quantity is to be raised, and the denominator indicates the root of this power which is to be extracted.

165. We have also seen that a^m may be divided by a^n, when m and n are whole numbers, by simply subtracting n from m, giving

$$\frac{a^m}{a^n} = a^{m-n} = a^p;$$

in which we have designated the excess of m over n by p.

Now, if n exceeds m, p becomes negative, and the exact division is impossible; but it has been agreed to retain the notation

$$\frac{a^m}{a^n} = a^{m-n} = a^{-p}.$$

But when $m < n$, in the fraction,

$$\frac{a^m}{a^n},$$

we may divide both terms by a^m, and we have

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} = \frac{1}{a^p};$$

hence, a^{-p} is equivalent to $\frac{1}{a^p}$, and both denote the reciprocal of a^p.

We have, then, from these principles, the following equivalent expressions, viz.:

$$\sqrt[n]{a} \quad \text{equivalent to} \quad a^{\frac{1}{n}}.$$

$$\sqrt[n]{a^m} \text{ or } (\sqrt[n]{a})^m \quad \text{“} \quad a^{\frac{m}{n}}.$$

$$\frac{1}{a^n} \quad \text{“} \quad a^{-n}.$$

$$\frac{1}{\sqrt[n]{a}} \text{ or } \sqrt[n]{\frac{1}{a}} \quad \text{“} \quad a^{-\frac{1}{n}}.$$

$$\frac{1}{\sqrt[n]{a^m}} \text{ or } \sqrt[n]{\frac{1}{a^m}} \quad \text{“} \quad a^{-\frac{m}{n}}.$$

166. It has been shown above that $\frac{1}{a^n} = a^{-n}$; if now we divide 1 by both members of this equation, we shall have, $a^n = \frac{1}{a^{-n}}$: hence we conclude that,

Any factor may be transferred from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent.

167. It may easily be shown that the rules for operating upon quantities when the *exponents* are positive whole numbers, are equally applicable when they are fractional or negative.

In the first place, it is plain that both numerator and denominator of the fractional exponent may be multiplied by the same quantity without altering the value of the expression, since by definition the m^{th} power of the m^{th} root of a quantity is equal to the quantity itself. This principle enables us to reduce quantities, having fractional exponents, to equivalent ones having a common denominator.

Let it be required to find the product of $a^{\frac{m}{n}}$ and $a^{\frac{r}{s}}$.

We have, $$a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{ms}{ns}} \times a^{\frac{nr}{ns}},$$

or (Art. 164), $$\sqrt[ns]{a^{ms}} \times \sqrt[ns]{a^{nr}} = \sqrt[ns]{a^{ms+nr}}:$$

This last result is equivalent to $a^{\frac{ms+nr}{ns}}$, hence,

$$a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{ms+nr}{ns}};$$

the same result that would have been obtained by the application of the rule for the multiplication of monomials, when the exponents are positive whole numbers.

If both exponents are negative, we shall have,

$$a^{-\frac{m}{n}} \times a^{-\frac{r}{s}} = \frac{1}{a^{\frac{m}{n}}} \times \frac{1}{a^{\frac{r}{s}}} = \frac{1}{a^{\frac{ms+nr}{ns}}} = a^{-\frac{ms+nr}{ns}}.$$

If one of the exponents is positive, and the other negative, we shall have,

$$a^{\frac{m}{n}} \times a^{-\frac{r}{s}} = a^{\frac{m}{n}} \times \frac{1}{a^{\frac{r}{s}}} = a^{\frac{ms}{ns}} \times \frac{1}{a^{\frac{nr}{ns}}};$$

whence, $\sqrt[ns]{a^{ms}} \times \sqrt[ns]{\dfrac{1}{a^{nr}}} = \sqrt[ns]{\dfrac{a^{ms}}{a^{nr}}} = \sqrt[ns]{a^{ms-nr}} = a^{\frac{ms-nr}{ns}};$

and finally, $a^{\frac{m}{n}} \times a^{-\frac{r}{s}} = a^{\frac{ms-nr}{ns}}.$

We have, therefore, for the multiplication of quantities when the exponents are negative or fractional, the same rule as when they are positive whole numbers, and consequently, the same rule for the formation of powers.

EXAMPLES.

1. $$a^{\frac{3}{4}}b^{-\frac{1}{2}}c^{-1} \times a^{2}b^{\frac{2}{3}}c^{\frac{3}{5}} = a^{\frac{11}{4}}b^{\frac{1}{6}}c^{-\frac{2}{5}}.$$

2. $$3a^{-2}b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}}b^{\frac{1}{2}}c^{2} = 6a^{-\frac{14}{5}}b^{\frac{7}{6}}c^{2}.$$

3. $$6a^{-\frac{1}{2}}b^{4}c^{-m} \times 5a^{\frac{1}{3}}b^{-5}c^{n} = 30a^{-\frac{1}{6}}b^{-1}c^{n-m}.$$

4. Find the square of $\frac{2}{3}a^{\frac{1}{3}}$.

We have, $\left(\frac{2}{3}a^{\frac{1}{3}}\right)^{2} = \left(\frac{2}{3}\right)^{2} \times a^{\frac{1}{3}\times 2} = \frac{4}{9}a^{\frac{2}{3}}.$

5. Find the cube of $\frac{1}{3}a^{\frac{1}{2}}$. *Ans.* $\frac{1}{27}a^{\frac{3}{2}}$.

168. Let it be required to divide $a^{\frac{m}{n}}$ by $a^{\frac{r}{s}}$. We shall have,

$$\frac{a^{\frac{m}{n}}}{a^{\frac{r}{s}}} = a^{\frac{m}{n}} \times a^{-\frac{r}{s}}, \quad \text{or (Art. 167),} \quad \frac{a^{\frac{m}{n}}}{a^{\frac{r}{s}}} = a^{\frac{ms-rn}{ns}}.$$

If both exponents are negative,

$$\frac{a^{-\frac{m}{n}}}{a^{-\frac{r}{s}}} = a^{-\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{rn-ms}{ns}},$$ by the last article.

If one exponent is negative,

$$\frac{a^{\frac{m}{n}}}{a^{-\frac{r}{s}}} = a^{\frac{m}{n}} \times a^{\frac{r}{s}} = a^{\frac{ms+rn}{ns}},$$ by the preceding article.

Hence, we see that the rule for the division of quantities, with fractional exponents, is the same as though the exponents were positive whole numbers; and consequently we have the same rule for the extraction of roots, as when the exponents are positive whole numbers.

EXAMPLES.

1. $$a^{\frac{2}{3}} \div a^{-\frac{3}{4}} = a^{\frac{2}{3}-(-\frac{3}{4})} = a^{\frac{17}{12}}.$$

2. $$a^{\frac{3}{4}} \div a^{\frac{4}{5}} = a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}}.$$

3. $$a^{\frac{2}{5}} \times b^{\frac{3}{4}} \div a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}.$$

4. Divide $32a^{\frac{1}{2}}b^6c^{\frac{5}{2}}$ by $8a^{\frac{1}{6}}b^5c^{-\frac{3}{2}}$. *Ans.* $4a^{\frac{1}{3}}bc^4$.

5. Divide $64a^9b^{\frac{7}{2}}c^{-\frac{3}{5}}$ by $32a^{-9}b^{-\frac{3}{2}}c^{-\frac{3}{5}}$. *Ans.* $2a^{18}b^5$.

6. $$\sqrt[3]{a^{\frac{2}{3}}} = a^{\frac{2}{9}};$$ 7. $$\sqrt[4]{a^{\frac{8}{11}}} = a^{\frac{2}{11}}.$$

8. $$\sqrt{a^{-\frac{3}{4}}} = a^{-\frac{3}{8}};$$ 9. $$\sqrt[3]{a^{\frac{3}{5}}b^{-2}} = a^{\frac{1}{5}}b^{-\frac{2}{3}}.$$

169. We see from the preceding discussion, that operations to be performed upon radicals, require no other rules than those previously established for quantities in which the exponents are entire. These operations are, therefore, reduced to simple operations upon fractions, with which we are already familiar.

GENERAL EXAMPLES.

1. Reduce $\dfrac{2\sqrt{2} \times (3)^{\frac{1}{3}}}{\frac{1}{2}\sqrt{2}}$ to its simplest terms.

Ans. $4\sqrt[3]{3}$.

2. Reduce $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}\,(3)^{\frac{1}{2}}}\right\}^4$ to its simplest terms.

Ans. $\dfrac{1}{384}\sqrt[3]{3}$.

3. Reduce $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{3\frac{1}{2}}}{2\sqrt{2}\,.\,(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$ to its simplest terms.

Ans. $\sqrt[4]{\dfrac{1}{6}\left(\dfrac{1}{8}\sqrt{6}+\sqrt{21}\right)}$.

4. What is the product of

$$a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}},\ \text{by}\ a^{\frac{1}{2}}-b^{\frac{1}{3}}.$$

Ans. a^3-b^2.

5. Divide $a^{\frac{7}{3}}-a^2b^{-\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}$, by $a^{\frac{1}{3}}-b^{-\frac{2}{3}}$.

Ans. a^2-b.

170. If we have an exponent which is a decimal fraction, as, for example, in the expression $10^{\cdot 301}$ from what has gone before the quantity is equal to $(10)^{\frac{301}{1000}}$, or to $\sqrt[1000]{(10)^{301}}$, the value of which it would be impossible to compute, by any process yet given, but which will hereafter be shown to be nearly equal to 2. In like manner, if the exponent is a radical, as $\sqrt{3}$, $\sqrt[3]{11}$, &c., we may treat the expression as though the exponents were fractional, since its values may be determined, to any degree of exactness, in decimal terms.

CHAPTER VII.

OF SERIES—ARITHMETICAL PROGRESSION—GEOMETRICAL PROPORTION AND PROGRESSION—RECURRING SERIES—BINOMIAL FORMULA—SUMMATION OF SERIES—PILING SHOT AND SHELLS.

171. A SERIES, in algebra, consists of an infinite number of terms following one another, each of which is derived from one or more of the preceding ones by a fixed law. This law is called the *law of the series.*

Arithmetical Progression.

172. An ARITHMETICAL PROGRESSION is a series, in which each term is derived from the preceding one by the addition of a constant quantity called the *common difference.*

If the common difference is *positive*, each term will be greater than the preceding one, and the progression is said to be *increasing.*

If the common difference is *negative*, each term will be less than the preceding one, and the progression is said to be *decreasing.*

Thus, . . . 1, 3, 5, 7, . . . &c., is an *increasing arithmetical progression*, in which the common difference is 2;
and . . . . . 19, 16, 13, 10, 7, . . . is a *decreasing arithmetical progression*, in which the common difference is -3.

173. When a certain number of terms of an arithmetical progression are considered, the first of these is called the *first term of the progression*, the last is called the *last term of the progression*, and both together are called the *extremes.* All the terms between the extremes are called *arithmetical means.* An arithmetical progression is often called a *progression by differences.*

174. Let d represent the common difference of the arithmetical progression,

$$a \,.\, b \,.\, c \,.\, e \,.\, f \,.\, g \,.\, h \,.\, k, \text{ \&c.},$$

which is written by placing a period between each two of the terms.

From the definition of a progression, it follows that,

$$b = a + d, \quad c = b + d = a + 2d, \quad e = c + d = a + 3d;$$

and, in general, any term of the series, is equal to *the first term plus as many times the common difference as there are preceding terms.*

Thus, let l be any term, and n the number which marks the place of it. Then, the number of preceding terms will be denoted by $n - 1$, and the expression for this *general term*, will be

$$l = a + (n - 1)\, d.$$

If d is positive, the progression will be increasing; hence,

In an increasing arithmetical progression, any term is equal to the first term, plus the product of the common difference by the number of preceding terms.

If we make $n = 1$, we have $l = a$; that is, there will be but one term.

If we make

$$n = 2, \quad \text{we have} \quad l = a + d;$$

that is, there will be two terms, and the second term is equal to the first plus the common difference.

EXAMPLES.

1. If $a = 3$ and $d = 2$, what is the 3d term? *Ans.* 7.

2. If $a = 5$ and $d = 4$, what is the 6th term? *Ans.* 25.

3. If $a = 7$ and $d = 5$. what is the 9th term? *Ans.* 47.

The formula,

$$l = a + (n - 1)\, d,$$

serves to find any term whatever, without determining those which precede it.

Thus to find the 50th term of the progression,

$$1 \,.\, 4 \,.\, 7 \,.\, 10 \,.\, 13 \,.\, 16 \,.\, 19, \ldots$$

we have, $l = 1 + 49 \times 3 = 148.$

And for the 60th term of the progression,

$$1 \,.\, 5 \,.\, 9. \,13 \,.\, 17 \,.\, 21 \,.\, 25, \ldots$$

we have, $l = 1 + 59 \times 4 = 237.$

174*. If d is negative, the progression is decreasing, and the formula becomes

$$l = a - (n-1)d; \quad \text{that is,}$$

Any term of a decreasing arithmetical progression, is equal to the first term plus the product of the common difference by the number of preceding terms.

EXAMPLES.

1. The first term of a decreasing progression is 60, and the common difference -3: what is the 20th term?

$$l = a - (n-1)d \quad \text{gives} \quad l = 60 - (20-1)3 = 60 - 57 = 3.$$

2. The first term is 90, the common difference -4: what is the 15th term? *Ans.* 34.

3. The first term is 100, and the common difference -2: what is the 40th term? *Ans.* 22.

175. If we take an arithmetical progression,

$$a \,.\, b \,.\, c \,.\, \ldots \ldots \,.\, i \,.\, k \,.\, l,$$

having n terms, and the common difference d, and designate the term which has p terms before it, by t, we shall have

$$t = a + pd \quad - \; - \; - \; - \; - \quad (1).$$

If we revert the order of terms of the progression, considering l as the first term, we shall have a new progression whose common difference is $-d$. The term of this new progression which has p terms before it, will evidently be the same as that which has p terms after it in the given progression, and if we represent that term by t', we shall have,

$$t' = l - pd \quad - \; - \; - \; - \quad (2).$$

Adding equations (1) and (2), member to member, we find

$$t + t' = a + l; \quad \text{hence,}$$

The sum of any two terms, at equal distances from the extremes of an arithmetical progression, is equal to the sum of the extremes.

176. If the sum of the terms of a progression be represented by S, and a new progression be formed, by reversing the order of the terms, we shall have

$$S = a + b + c + \ldots + i + k + l,$$

$$S = l + k + i + \ldots + c + b + a.$$

Adding these equations, member to member, we get

$$2S = (a + l) + (b + k) + (c + i) \ldots + (i + c) + (k + b) + (l + a);$$

and, since all the sums, $a + l$, $b + k$, $c + i \ldots$ are equal to each other, and their number equal to n, the number of terms in the progression, we have

$$2S = (a + l)\,n, \quad \text{or} \quad S = \left(\frac{a + l}{2}\right) n; \quad \text{that is,}$$

The sum of the terms of an arithmetical progression is equal to half the sum of the two extremes multiplied by the number of terms.

EXAMPLES.

1. The extremes are 2 and 16, and the number of terms 8: what is the sum of the series?

$$S = \left(\frac{a + l}{2}\right) \times n, \quad \text{gives} \quad S = \frac{2 + 16}{2} \times 8 = 72.$$

2. The extremes are 3 and 27, and the number of terms 12: what is the sum of the series? *Ans.* 180.

3. The extremes are 4 and 20, and the number of terms 10: what is the sum of the series? *Ans.* 120.

4. The extremes are 8 and 80, and the number of terms 10: what is the sum of the series? *Ans.* 440.

The formulas

$$l = a + (n - 1)\,d \quad \text{and} \quad S = \left(\frac{a + l}{2}\right) \times n,$$

contain five quantities, a, d, n, l, and S, and consequently give rise to the following general problem, viz.:

Any three of these five quantities being given, to determine the other two.

This general problem gives rise to the ten following cases:—

No.	Given.	Unknown.	Values of the Unknown Quantities.
1	a, d, n	l, S	$l=a+(n-1)d;\ S=\frac{1}{2}n[2a+(n-1)d]$.
2	a, d, l	n, S	$n=\frac{l-a}{d}+1;\ S=\frac{(l+a)(l-a+d)}{2d}$.
3	a, d, S	n, l	$n=\frac{d-2a\pm\sqrt{(d-2a)^2+8dS}}{2d};\ l=a+(n-1)d$.
4	a, n, l	S, d	$S=\frac{1}{2}n(a+l);\ d=\frac{l-a}{n-1}$.
5	a, n, S	d, l	$d=\frac{2(S-an)}{n(n-1)};\ l=\frac{2S}{n}-a$.
6	a, l, S	n, d	$n=\frac{2S}{a+l};\ d=\frac{(l+a)(l-a)}{2S-(l+a)}$.
7	d, n, l	a, S	$a=l-(n-1)d;\ S=\frac{1}{2}n[2l-(n-1)d]$.
8	d, n, S	a, l	$a=\frac{2S-n(n-1)d}{2n};\ l=\frac{2S+n(n-1)d}{2n}$.
9	d, l, S	n, a	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d};\ a=l-(n-1)d$.
10	n, l, S	a, d	$a=\frac{2S}{n}-l;\ d=\frac{2(nl-S)}{n(n-1)}$.

177. From the formula

$$l=a+(n-1)d,$$

we have,
$$a=l-(n-1)d;$$
that is,

The first term of an increasing arithmetical progression, is equal to any following term, minus the product of the common difference by the number of preceding terms.

178. From the same formula, we also find

$$d=\frac{l-a}{n-1};$$
that is,

In any arithmetical progression, the common difference is equal to the last term minus the first term, divided by the number of terms less one.

If the last term is less than the first, the common difference will be negative, as it should be.

EXAMPLES.

1. The first term of a progression is 4 the last term 16, and the number of terms considered 5: what is the common difference?

The formula

$$d = \frac{l - a}{n - 1} \quad \text{gives} \quad d = \frac{16 - 4}{4} = 3.$$

2. The first term of a progression is 22, the last term 4, and the number of terms considered 10: what is the common difference? *Ans.* -2.

179. By the aid of the last principle deduced, we can solve the following problem, viz.:

To find a number m of arithmetical means between two given numbers a and b.

To solve this problem, it is first necessary to find the common difference. Now, we may regard a as the first term of an arithmetical progression, b as the last term, and the required means as intermediate terms. The number of terms considered, of this progression, will be expressed by $m + 2$.

Now, by substituting in the above formula, b for l, and $m + 2$ for n, it becomes

$$d = \frac{b - a}{m + 2 - 1}, \quad \text{or} \quad d = \frac{b - a}{m + 1},$$

that is, *the common difference* of the required progression is obtained by dividing the difference between the last and first terms by one more than the required number of means.

Having obtained the common difference, form the second term of the progression, or the *first arithmetical mean*, by adding d, or $\frac{b-a}{m+1}$, to the first term a. The *second mean* is obtained by augmenting the first by d, &c.

EXAMPLES.

1. Find 3 arithmetical means between 2 and 18. The formula

$$d = \frac{b-a}{m+1}, \quad \text{gives} \quad d = \frac{18-2}{4} = 4;$$

hence, the progression is

$$2 \,.\, 6 \,.\, 10 \,.\, 14 \,.\, 18.$$

2. Find 12 arithmetical means between 77 and 12. The formula

$$\frac{b-a}{m+1}, \quad \text{gives} \quad d = \frac{12-77}{13} = -5;$$

hence, the progression is

$$77 \,.\, 72 \,.\, 67 \,.\, 62 \ldots\ldots 22 \,.\, 17 \,.\, 12.$$

3. Find 9 arithmetical means and the series, between 75 and 5.

Ans. Progression 75 . 68 . 61 . . . . . 26 . 19 . 12 . 5.

180. If the same number of arithmetical means be inserted between the terms of a progression, taken two and two, these terms, and the arithmetical means together, will form one and the same progression.

For, let $a \,.\, b \,.\, c \,.\, e \,.\, f \ldots\ldots$ be the proposed progression, and m the number of means to be inserted between a and b, b and c, c and e. . . . .

From what has just been said, the common difference of each partial progression will be expressed by

$$\frac{b-a}{m+1}, \quad \frac{c-b}{m+1}, \quad \frac{e-c}{m+1} \cdots\cdots$$

which are equal to each other, since, a, b, c, . . . are in progression: therefore, the common difference is the same in each

of the partial progressions; and since the *last term* of the first, forms the *first term* of the second, &c., we may conclude that all of these partial progressions form a single progression.

GENERAL EXAMPLES.

1. Find the sum of the first fifty terms of the progression

$$2 . 9 . 16 . 23 \ldots$$

For the 50th term, we have

$$l = 2 + 49 \times 7 = 345.$$

Hence, $\quad S = (2 + 345) \times \frac{50}{2} = 347 \times 25 = 8675.$

2. Find the 100th term of the series $2 . 9 . 16 . 23 \ldots$

Ans. 695.

3. Find the sum of 100 terms of the series $1 . 3 . 5 . 7 . 9 \ldots$

Ans. 10000.

4. The greatest term considered is 70, the common difference 3, and the number of terms 21: what is the least term and the sum of the terms?

Ans. Least term 10; sum of terms 840.

5. The first term of a decreasing arithmetical progression is 10, the common difference is $-\frac{1}{3}$, and the number of terms 21: required the sum of the terms. *Ans.* 140.

6. In a progression by differences, having given the common difference 6, the last term 185, and the sum of the terms 2945: find the first term, and the number of terms.

Ans. First term $= 5$; number of terms 31.

7. Find 9 arithmetical means between each antecedent and consequent of the progression $2 . 5 . 8 . 11 . 14 \ldots$

Ans. $d = 0.3$.

8. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2, the third 3, and so on to the n^{th}, which contains n. In other words,

find the expression for the sum of the natural numbers 1, 2, 3, . . . from 1 to n, inclusively.

Ans. $S = \frac{n(n+1)}{2}$.

9. Find the sum of the first n terms of the progression of uneven numbers 1, 3, 5, 7, 9 . . . *Ans.* $S = n^2$.

10. One hundred stones being placed on the ground, in a straight line, at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket, placed at two yards from the first stone?

Ans. 11 miles 840 yards.

Of Ratio and Geometrical Proportion.

181. The Ratio of one quantity to another, is the quotient which arises from dividing the second by the first. Thus, the ratio of a to b, is $\frac{b}{a}$.

182. *Two* quantities are said to be proportional, or in proportion, when their ratio remains the same, while the quantities themselves undergo changes of value. Thus, if the ratio of a to b remains the same, while a and b undergo changes of value, then a is said to be proportional to b.

183. *Four* quantities are in *proportion*, when the ratio of the first to the second, is equal to the ratio of the third to the fourth.

Thus, if

$$\frac{b}{a} = \frac{d}{c},$$

the quantities a, b, c and d, are said to be in proportion. We generally express that these quantities are proportional by writing them as follows:

$$a : b : : c : d.$$

This algebraic expression is read, a is to b, as is to d, and is called a *proportion.*

184. The quantities compared, are called *terms* of the proportion.

The first and fourth terms are called *the extremes*, the second and third are called *the means;* the first and third are called *antecedents*, the second and fourth are called *consequents*, and the fourth is said to be a *fourth proportional* to the other three.

If the second and third terms are the same, either of these is said to be a *mean proportional* between the other two. Thus, in the proportion

$$a : b : : b : c,$$

b is a mean proportional between a and c, and c is said to be a third proportional to a and b.

185. Two quantities are reciprocally proportional when one is proportional to the reciprocal of the other.

Geometrical Progression.

186. A GEOMETRICAL PROGRESSION is a series of terms, each of which is derived from the preceding one, by multiplying it by a constant quantity, called the *ratio* of the progression.

If the ratio is greater than 1, each term is greater than the preceding one, and the progression is said to be *increasing*.

If the ratio is less than 1, each term is less than the preceding one, and the progression is said to be *decreasing.*

Thus,

. . . 3, 6, 12, 24, . . . &c., is an increasing progression.

. . . $16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$, . . . is a decreasing progression

It may be observed that a geometrical progression is a continued proportion in which each term is a mean proportional between the preceding and succeeding terms.

187. Let r designate the ratio of a geometrical progression,

$$a : b : c : d, \ldots . \&c.$$

We deduce from the definition of a progression the following equations:

$$b = ar, \quad c = br = ar^2, \quad d = cr = ar^3, \quad e = dr = ar^4 \quad \ldots;$$

and, in general, any term n, that is, one which has $n-1$ terms before it, is expressed by ar^{n-1}.

Let l be this term; we have the formula

$$l = ar^{n-1},$$

by means of which we can obtain any term without being obliged to find all the terms which precede it. That is,

Any term of a geometrical progression is equal to the first term multiplied by the ratio raised to a power whose exponent denotes the number of preceding terms.

EXAMPLES.

1. Find the 5th term of the progression

$$2 : 4 : 8 : 16, \text{ \&c.},$$

in which the first term is 2, and the common ratio 2.

$$5\text{th term} = 2 \times 2^4 = 2 \times 16 = 32.$$

2. Find the 8th term of the progression

$$2 : 6 : 18 : 54 \ldots$$

$$8\text{th term} = 2 \times 3^7 = 2 \times 2187 = 4374.$$

3. Find the 12th term of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \ldots$$

$$12\text{th term} = 64\left(\frac{1}{4}\right)^{11} = \frac{4^3}{4^{11}} = \frac{1}{4^8} = \frac{1}{65536}.$$

188. We will now explain the method of determining the sum of n terms of the progression

$$a : b : c : d : e : f : \ldots : i : k : l,$$

of which the ratio is r.

If we denote the sum of the series by S, and the n^{th} term by l, we shall have

$$S = a + ar + ar^2 \ldots . + ar^{n-2} + ar^{n-1}.$$

If we multiply both members by r, we have

$$Sr = ar + ar^2 + ar^3 \ldots + ar^{n-1} + ar^n;$$

and by subtracting the first equation from the second, member from member,

$$Sr - S = ar^n - a, \quad \text{whence,} \quad S = \frac{ar^n - a}{r - 1};$$

substituting for ar^n, its value lr, we have

$$S = \frac{lr - a}{r - 1}; \quad \text{that is,}$$

To obtain the sum of any number of terms of a progression by quotients,

Multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by 1.

EXAMPLES.

1. Find the sum of eight terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots : 4374.$$

$$S = \frac{lr - a}{r - 1} = \frac{13122 - 2}{2} = 6560.$$

2. Find the sum of five terms of the progression

$$2 : 4 : 8 : 16 : 32; \ldots .$$

$$S = \frac{lr - a}{r - 1} = \frac{64 - 2}{1} = 62.$$

3. Find the sum of ten terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots 2 \times 3^9 = 39366.$$

Ans. 59048.

4. What debt may be discharged in a year, or twelve months, by paying \$1 the first month, \$2 the second month, \$4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. Debt, \$4095; last payment, \$2048.

5. A gentleman married his daughter on New-Year's day, and gave her husband 1*s*. toward her portion, and was to double it on the first day of every month during the year: what was her portion?

Ans. £204 15*s*.

6. A man bought 10 bushels of wheat on the condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on to the last: what did he pay for the last bushel, and for the ten bushels?

Ans. Last bushel, \$196 83; total cost, \$295, 24.

189. When the progression is decreasing, we have $r < 1$ and $l < a$; the above formula for the sum is then written under the form

$$S = \frac{a - lr}{1 - r}$$

in order that both terms of the fraction may be positive.

By substituting ar^{n-1} for l, in the expression for S,

$$S = \frac{ar^n - a}{r - 1}, \quad \text{or} \quad S = \frac{a - ar^n}{1 - r}.$$

EXAMPLES.

1. Find the sum of the first five terms of the progression

$$32 : 16 : 8 : 4 : 2.$$

$$S = \frac{a - lr}{1 - r} = \frac{32 - 2 \times \frac{1}{2}}{\frac{1}{2}} = \frac{31}{\frac{1}{2}} = 62.$$

2. Find the sum of the first twelve terms of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \; . \; . \; . \; . : \frac{1}{65536}.$$

$$S = \frac{a - lr}{1 - r} = \frac{64 - \frac{1}{65536} \times \frac{1}{4}}{\frac{3}{4}} = \frac{256 - \frac{1}{65536}}{3} = 85 + \frac{65535}{196608}.$$

We perceive that the principal difficulty consists in obtaining the numerical value of the last term, a tedious operation, even when the number of terms is not very great.

190. If in the formula

$$S = \frac{a(r^n - 1)}{r - 1},$$

we make $r = 1$, it reduces to

$$S = \frac{0}{0}.$$

This result sometimes indicates indetermination; but it often arises from the existence of a common factor in both numerator and denominator of the fraction, which factor becomes 0, in consequence of a particular supposition.

Such is the fact in the present case, since both terms of the fraction contain the factor $r - 1$, which becomes 0, for the particular supposition $r = 1$.

If we divide both terms of the fraction by this common factor, we shall find (Art. 60),

$$S = ar^{n-1} + ar^{n-2} + ar^{n-3} + \ldots + ar + a,$$

in which, if we make $r = 1$, we get

$$S = a + a + a + a + \ldots\ldots + a = na.$$

We ought to have obtained this result; for, under the supposition made, each term of the progression became equal to a, and since there are n of them, their sum should be na.

191. From the two formulas

$$l = ar^{n-1}, \quad \text{and} \quad S = \frac{lr - a}{r - 1},$$

several properties may be deduced. We shall consider only some of the most important.

The first formula gives

$$r^{n-1} = \frac{l}{a}, \quad \text{whence} \quad r = \sqrt[n-1]{\frac{l}{a}}.$$

The expression

$$r = \sqrt[n-1]{\frac{l}{a}},$$

furnishes the means for resolving the following problem, viz.

To find m geometrical means between two given numbers a and b; that is, *to find a number m of means, which will form with a and b, considered as extremes, a geometrical progression.*

To find this series, it is only necessary to know the *ratio.* Now, the required number of means being m, the total number of terms considered, will be equal to $m+2$. Moreover, we have $l=b$; therefore, the value of r becomes

$$r = \sqrt[m+1]{\frac{b}{a}}; \quad \text{that is,}$$

To find the ratio, divide the second of the given numbers by the first; then extract that root of the quotient whose index is one greater than the required number of means:

Hence the progression is

$$a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m+1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} : \ldots b.$$

EXAMPLES.

1. To insert six geometrical means between the numbers 3 and 384, we make $m=6$, whence from the formula,

$$r = \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2;$$

hence, we deduce the progression

$$3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

2. Insert four geometrical means between the numbers 2 and 486. The progression is

$$2 : 6 : 18 : 54 : 162 : 486.$$

REMARK.—When the same number of geometrical means are inserted between each two of the terms of a geometrical progression, all the progressions thus formed will, when taken together, constitute a single progression.

Progressions having an infinite number of terms.

192. Let there be the decreasing progression

$$a : b : c : d : e : f : \ldots,$$

containing an infinite number of terms. The formula

$$S = \frac{a - ar^n}{1 - r},$$

which expresses the sum of n terms, can be put under the form

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now, since the progression is decreasing, r is a proper fraction, and r^n is also a fraction, which diminishes as n increases. Therefore, the greater the number of terms we take, the more will $\frac{a}{1-r} \times r^n$ diminish, and consequently, the nearer will the sum of these terms approximate to an equality with the first part of S; that is, to $\frac{a}{1-r}$. Finally, when n is taken greater than any assignable number, or when

$$n = \infty, \quad \text{then} \quad \frac{a}{1-r} \times r^n$$

will be less than any assignable number, or will become equal to 0; and the expression $\frac{a}{1-r}$ will represent the true value of the sum of all the terms of the series. Hence,

The sum of the terms of a decreasing progression, in which the number of terms is infinite, is

$$S = \frac{a}{1-r}.$$

This is, properly speaking, the *limit* to which the *partial sums* approach, as we take a greater number of terms of the progression. The number of terms may be taken so great as to make the difference between the sum, and $\frac{a}{1-r}$, as small as we please, and the difference will only become *zero* when the number of terms taken is infinite.

EXAMPLES.

1. Find the sum of

$$1 : \frac{1}{3} : \frac{1}{9} : \frac{1}{27} : \frac{1}{81} : \text{\&c.}.$$

We have for the sum of the terms,

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2. Again, take the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \&c. \; . \; . \; .$$

We have $$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

What is the error, in each example for $n = 4$, $n = 5$, $n = 6$?

Indeterminate Co-efficients.

193. An Identical Equation is one which is satisfied for any values that may be assigned to one or more of the quantities which enter it. It differs materially from an ordinary equation.

The latter, when it contains but one unknown quantity, can only be satisfied for a limited number of values of that quantity, whilst the former is satisfied for any value whatever of the indeterminate quantity which enters it.

It differs also from the indeterminate equation. Thus, if in the ordinary equation

$$ax + by + cz + d = 0$$

values be assigned to x and y at pleasure, and corresponding values of z be deduced from the equation, these values taken together will satisfy the equation, and an infinite number of sets of values may be found which will satisfy it (Art. 88).

But if in the equation

$$ax + by + cz + d = 0,$$

we impose the condition that it shall be satisfied for any values of x, y and z, taken at pleasure, it is then called an *identical equation.*

194. A quantity is *indeterminate* when it admits of an infinite number of values.

Let us assume the identical equation,

$$A + Bx + Cx^2 + Dx^3 + \&c = 0 \quad . \; . \; . \; . \; . \quad (1),$$

in which the co-efficients, A, B, C, D, &c., are entirely independent of x.

If we make $x = 0$ in equation (1) all the terms containing x reduce to 0, and we find

$$A = 0.$$

Substituting this value of A in equation (1), and factoring, it becomes,

$$x\,(B + Cx + Dx^2 + \text{\&c.,}) = 0 \quad - \; - \; - \; - \; - \quad (2),$$

which may be satisfied by placing $x = 0$, or by placing

$$B + Cx + Dx^2 + \text{\&c.} = 0 \quad - \; - \quad - \; - \; - \quad (3).$$

The first supposition gives a common equation, satisfied only for $x = 0$. Hence, equation (2) can only be an identical equation under a supposition which makes equation (3) an identical equation.

If, now, we make $x = 0$ in equation (3), all the terms containing x will reduce to 0, and we find

$$B = 0.$$

Substituting this value of B in equation (3), and factoring, we get

$$x\,(C + Dx + \text{\&c.}) = 0 \quad - \; - \; - \; - \; - \; - \; - \; - \; - \quad (4).$$

In the same manner as before, we may show that $C = 0$, and so we may prove in succession that each of the co-efficients D, E, &c., is separately equal to 0: hence,

In every identical equation, either member of which is 0, *involving a single indeterminate quantity, the co-efficients of the different powers of this quantity are separately equal to* 0.

195. Let us next assume the identical equation

$$a + bx + cx^2 + \text{\&c.} = a' + b'x + c'x^2 + \text{\&c.}$$

By transposing all the terms into the first member, it may be placed under the form

$$(a - a') + (b - b')\,x + (c - c')\,x^2 + \text{\&c.} = 0.$$

Now, from the principle just demonstrated,

$$a - a' = 0, \quad b - b' = 0, \quad c - c' = 0, \quad \text{\&c., \&c.;} \; = 0$$

whence $\quad a = a',\; b = b' \quad c = c',$ &c., &c.; that is,

In an identical equation containing but one indeterminate quantity, the co-efficients of the like powers of that quantity in the two members, are equal to each other.

196. We may extend the principles just deduced to identical equations containing any number of indeterminate quantities.

For, let us assume that the equation

$$a + bx + b'y + b''z + \&c. + cx^2 + c'y^2 + c''z^2 + \&c. + dx^3 + d'y^3 + \&c. = 0 \quad - \quad - \quad - \quad (1),$$

is satisfied independently of any values that may be assigned to x, y, z, &c. If we make all the indeterminate quantities xcept x equal to 0, equation (1), will reduce to

$$a + bx + cx^2 + dx^3 + \&c. = 0\,;$$

whence, from the principle of article 194,

$$a = 0, \quad b = 0, \quad c = 0, \quad d = 0, \quad \&c. = 0$$

If, now, we make all the arbitrary quantities except y equal to 0, equation (1) reduces to,

$$a + b'y + c'y^2 + d'y^3 + \&c. = 0\,;$$

whence, as before,

$$a = 0, \quad b' = 0, \quad c' = 0, \quad d' = 0, \quad \&c. = 0$$

and similarly we have

$$b'' = 0, \quad c'' = 0, \quad \&c. =$$

The principle here developed is called the *principle of indeterminate co-efficients*, not because the co-efficients are really indeterminate, for we have shown that they are separately equal to 0, but because they are co-efficients of indeterminate quantities.

197. The principle of Indeterminate Co-efficients is much used in developing algebraic expressions into series.

For example, let us endeavor to develop the expression,

$$\frac{a}{a' + b'x},$$

into a series arranged according to the ascending powers of x.

Let us assume a development of the proposed form,

$$\frac{a}{a' + b'x} = P + Qx + Rx^2 + Sx^3 + \&c. \quad - \; - \; - \quad (1),$$

in which P, Q, R, &c., are independent of x, and depend upon a, a' and b' for their values. It is now required to find such values for P, Q, R, &c., as will make the development a true one for all values of x.

By clearing of fractions and transposing all the terms into the first member, we have

$$\left.\begin{array}{r} Pa' \\ -a \end{array}\right| \left.\begin{array}{r} +\, Qa' \\ +\, Pb' \end{array}\right| x \left.\begin{array}{r} +\, Ra' \\ +\, Qb' \end{array}\right| x^2 \begin{array}{l} +\ \&c. = 0. \\ \quad \&c. \end{array}$$

Since this equation is true for all values of x, it is identical, and from the principle of Art. 194, we have

$Pa' - a = 0$, $Qa' + Pb' = 0$, $Ra' + Qb' = 0$, &c., &c.; whence,

$$P = \frac{a}{a'}, \quad Q = -\frac{Pb'}{a'} = -\frac{ab'}{a'^2}, \quad R = -\frac{Qb'}{a'} = \frac{ab'^2}{a'^3}, \ \&c., \&c.$$

Substituting these values of P, Q, R, &c., in equation (1), it becomes

$$\frac{a}{a' + b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \&c. \quad - \; - \quad (2).$$

Since we may pursue the same course of reasoning upon any like expression, we have for developing an algebraic expression into a series, the following

RULE.

I. *Place the given expression equal to a development of the form* $P + Qx + Rx^2 + $ *&c., clear the resulting equation of fractions, and transpose all of the terms into the first member of the equation.*

II. *Then place the co-efficients of the different powers of the letter, with reference to which the series is arranged, separately equal to* 0, *and from these equations find the values of* P, Q, R, *&c.*

III. *Having found these values, substitute them for* P, Q, R, *&c., in the assumed development, and the result will be the development required.*

EXAMPLES.

1. Develop $\dfrac{a}{a-x}$ into a series.

Ans. $1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\dfrac{x^3}{a^3}+$ &c.

2. Develop $\dfrac{1}{(a-x)^2}$ into a series.

Ans. $\dfrac{1}{a^2}+\dfrac{2x}{a^3}+\dfrac{3x^2}{a^4}+\dfrac{4x^3}{a^5}+$ &c.

3. Develop $\dfrac{1+2x}{1-3x}$ into a series.

Ans. $1+5x+15x^2+45x^3+135x^4+$ &c.

198. We have hitherto supposed the series to be arranged according to the ascending powers of the unknown quantity, commencing with the 0 power, but all expressions cannot be developed according to this law. In such cases, the application of the rule gives rise to some absurdity.

For example, if we apply the rule to develop $\dfrac{1}{3x-x^2}$, we shall have,

$$\frac{1}{3x-x^2}=P+Qx+Rx^2+\text{\&c.} \quad - - - \quad (1).$$

Clearing of fractions, and transposing,

$$\left.\begin{array}{r}-1+3Px+3Q\\ -P\end{array}\right|x^2+\text{\&c.}=0;$$

Whence, by the rule,

$$-1=0, \quad 3P=0, \quad 3Q-P=0, \text{ \&c.}$$

Now, the first equation is absurd, since -1 cannot equal 0. Hence, we conclude that the expression cannot be developed according to the ascending powers of x, beginning at x^0.

We may, however, write the expression under the form $\dfrac{1}{x}\times\dfrac{1}{3-x}$, and by the application of the rule, develop the factor $\dfrac{1}{3-x}$, which gives

$$\frac{1}{3-x}=\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+\frac{1}{81}x^3+\text{\&c.};$$

whence, by substitution,

$$\frac{1}{3x - x^2} = \frac{1}{3x} + \frac{1}{9}x^0 + \frac{1}{27}x + \frac{1}{81}x^2 + \&c.$$

Since $\frac{1}{3x}$ is equal to $3x^{-1}$ (Art. 166), we see that the *true* development contains a term with a negative exponent, and the supposition made in equation (1) ought to have failed.

Recurring Series.

199. The development of fractions of the form $\frac{a}{a' + b'x}$, &c., gives rise to the consideration of a kind of series, called *recurring series.*

A RECURRING SERIES is one in which any term is equal to the algebraic sum of the products obtained by multiplying one or more of the preceding terms by certain fixed quantities.

These fixed quantities, taken in their proper order, constitute what is called the *scale of the series.*

200. If we examine the development

$$\frac{a}{a' + b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \&c.,$$

we shall see, that each term is formed by multiplying the preceding one by $-\frac{b'}{a'}x$. This is called a recurring series of the *first order*, because the scale of the series contains but one term.

The expression $-\frac{b'}{a'}x$ is the *scale of the series*, and the expression $-\frac{b'}{a'}$ is called the *scale of the co-efficients.*

It may be remarked, that a geometrical progression is a recurring series of the first order.

201. Let it be required to develop the expression

$$\frac{a + bx}{a' + b'x + c'x^2} \text{ into a series.}$$

Assume $$\frac{a+bx}{a'+b'x+c'x^2} = P + Qx + Rx^2 + Sx^3 + \&c.$$

Clearing of fractions, and transposing, we get

$$\left.\begin{array}{r} Pa' \\ -a \end{array}\right| \left.\begin{array}{r} +Qa' \\ +Pb' \\ -b \end{array}\right| x \left.\begin{array}{r} +Ra' \\ +Qb' \\ +Pc' \end{array}\right| x^2 \left.\begin{array}{r} +Sa' \\ +Rb' \\ +Qc' \end{array}\right| x^3 + \&c. = 0.$$

Therefore, we have

$$Pa' - a = 0, \quad \text{or,} \quad P = \frac{a}{a'};$$

$$Qa' + Pb' - b = 0, \quad \text{or,} \quad Q = -\frac{b'}{a'}P + \frac{b}{a'};$$

$$Ra' + Qb' + Pc' = 0, \quad \text{or,} \quad R = -\frac{b'}{a'}Q - \frac{c'}{a'}P;$$

$$Sa' + Rb' + Qc' = 0, \quad \text{or,} \quad S = -\frac{b'}{a'}R - \frac{c'}{a'}Q;$$

&c., &c., &c., &c.;

from which we see that, commencing at the third, each co-efficient is formed by multiplying the two which precede it, respectively, by $-\frac{b'}{a'}$ and $-\frac{c'}{a'}$, viz., that which immediately precedes the required co-efficient by $-\frac{b'}{a'}$, that which precedes it two terms by $-\frac{c'}{a'}$, and taking the algebraic sum of the products. Hence,

$$\left(-\frac{b'}{a'}, \quad -\frac{c'}{a'}\right)$$

is the *scale of the co-efficients.*

From this law of formation of the co-efficients, it follows that the third term, and every succeeding one, is formed by multiplying the one that next precedes it by $-\frac{b'}{a'}x$, and the second preceding one by $-\frac{c'}{a'}x^2$, and then taking the algebraic sum of these products: hence,

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2\right)$$

is the *scales of the series.*

This scale contains two terms, and the series is called a recurring series of the *second order.* In general, the order of a recurring series is denoted by the number of terms in the scale of the series.

The development of the fraction

$$\frac{a + bx + cx^2}{a' + b'x + c'x^2 + d'x^3},$$

gives rise to a recurring series of the third order, the scale of which is,

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2, \quad -\frac{d'}{a'}x^3\right);$$

and, in general, the development of

$$\frac{a + bx + cx^2 + \ldots kx^{n-1}}{a' + b'x + c'x^2 + \ldots k'x^n},$$

gives a recurring series *of the* n^{th} *order*, the scale of which is

$$\left(-\frac{b'}{a'}x, \quad -\frac{c'}{a'}x^2 \ldots, \quad -\frac{k'}{a'}x^n\right)$$

General demonstration of the Binomial Theorem.

202. It has been shown (Art. 60), that any expression of the form $z^m - y^m$, is exactly divisible by $z - y$, when m is a positive whole number, giving,

$$\frac{z^m - y^m}{z - y} = z^{m-1} + z^{m-2}y + z^{m-3}y^2 + \ldots . + y^{m-1}.$$

The number of terms in the quotient is equal to m, and if we suppose $z = y$, each term will become z^{m-1}; hence,

$$\left(\frac{z^m - y^m}{z - y}\right)_{y = z} = mz^{m-1}.$$

The notation employed in the first member, simply indicates what the quantity within the parenthesis becomes when we make $y = z$.

We now propose to show that this *form* is true when m is fractional and when it is negative.

First, suppose m fractional, and equal to $\frac{p}{q}$.

Make $z^{\frac{1}{q}} = v,$ whence $z^{\frac{p}{q}} = v^p$ and $z = v^q;$

and $y^{\frac{1}{q}} = u,$ whence $y^{\frac{p}{q}} = u^p$ and $y = u^q.$

hence, $$\frac{z^{\frac{p}{q}} - y^{\frac{p}{q}}}{z - y} = \frac{v^p - u^p}{v^q - u^q} = \frac{\dfrac{v^p - u^p}{v - u}}{\dfrac{v^q - u^q}{v - u}}.$$

If now, we suppose $y = z$, we have $v = u$, and since p and q are positive whole numbers, we have

$$\left\{\frac{z^{\frac{p}{q}} - y^{\frac{p}{q}}}{z - y}\right\}_{y=z} = \frac{\left(\dfrac{v^p - u^p}{v - u}\right)_{v=u}}{\left(\dfrac{v^q - u^q}{v - u}\right)_{v=u}} = \frac{pv^{p-1}}{qv^{q-1}} = \frac{p}{q}v^{p-q} = \frac{p}{q}z^{\frac{p}{q}-1}$$

Second, suppose m negative, and either entire or fractional.

By observing that

$$-z^{-m}y^{-m} \times (z^m - y^m) = z^{-m} - y^{-m},$$

we have,

$$\frac{z^{-m} - y^{-m}}{z - y} = -z^{-m}y^{-m} \times \frac{z^m - y^m}{z - y}.$$

If, now, we make the supposition that $y = z$, the first factor of the second member reduces to $-z^{-2m}$, and the second factor, from the principles just demonstrated, reduces to mz^{m-1}; hence,

$$\left(\frac{z^{-m} - y^{-m}}{z - y}\right)_{y=z} = -mz^{-m-1}.$$

We conclude, therefore, that the *form is general.*

203. By the aid of the principles demonstrated in the last article, we are able to deduce a formula for the development of

$$(x + a)^m,$$

when the exponent m *is positive or negative, entire or fractional*

Let us assume the equation,

$$(1 + z)^m = P + Qz + Rz^2 + Sz^3 + \&c. \quad - \quad - \quad (1),$$

in which, P, Q, R, &c., are independent of z, and depend upon 1 and m for their values. It is required to find such values for them as will make the assumed development true for every possible value of z.

If, in equation (1) we make $z = 0$, we have

$$P = 1.$$

Substituting this value for P, equation (1) becomes,

$$(1 + z)^m = 1 + Qz + Rz^2 + Sz^3 + \&c. \quad - - - \quad (2).$$

Equation (2) being true for all values of z, let us make $z = y$; whence,

$$(1 + y)^m = 1 + Qy + Ry^2 + Sy^3 + \&c. \quad - - - \quad (3).$$

Subtracting equation (3) from (2), member from member, and dividing the first member by $(1 + z) - (1 + y)$, and the second member by its equal $z - y$, we have,

$$\frac{(1+z)^m - (1+y)^m}{(1+z) - (1+y)} = Q\frac{z-y}{z-y} + R\frac{z^2 - y^2}{z-y} + S\frac{z^3 - y^3}{z-y} + \&c. - (4).$$

If, now, we make $1 + z = 1 + y$, whence $z = y$, the first member of equation (4), from previous principles, becomes $m(1 + z)^{m-1}$, and the quotients in the second member become respectively,

$$\left(\frac{z-y}{z-y}\right)_{y=z} = 1, \left(\frac{z^2-y^2}{z-y}\right)_{y=z} = 2z, \left(\frac{z^3-y^3}{z-y}\right)_{y=z} = 3z^2, \&c. \&c.$$

Substituting these results in equation (4) we have,

$$m(1 + z)^{m-1} = Q + 2Rz + 3Sz^2 + 4Tz^3 + \&c. \quad - - - \quad (5).$$

Multiplying both members of equation (5) by $(1 + z)$, we find,

$$m(1 + z)^m = \begin{array}{l|l|l|l} Q + 2R & z + 3S & z^2 + 4T & z^3 + \&c. \\ \;\;\; + \;\; Q & \;\; + 2R & \;\; + 3S & \end{array} \quad - - - \quad (6).$$

If we multiply both members of equation (2) by m, we have

$$m(1 + z)^m = m + mQz + mRz^2 + mSz^3 + mTz^4 + \&c. \quad - - - \quad (7).$$

The second members of equations (6) and (7) are equal to each other, since the first members are the same; hence, we have the equation,

$$m + mQz + mRz^2 + mSz^3 + \&c. = \begin{array}{l|l|l|l} Q + 2R & z + 3S & z^2 + 4T & z^3 + \&c \\ \;\; + \;\; Q & + 2R & + 3S & \end{array} - (8)$$

This equation being identical, we have, (Art. 195),

$$Q = m, \quad \text{- - or, - -} \quad Q = \frac{m}{1}.$$

$$2R + Q = mQ, \quad \text{- or, - -} \quad R = \frac{m(m-1)}{1 \;.\; 2};$$

$$3S + 2R = mR, \quad \text{- or, - -} \quad S = \frac{m(m-1)(m-2)}{1 \;.\; 2 \;.\; 3};$$

$$4T + 3S = mS, \quad \text{- or, -} \quad T = \frac{m(m-1)(m-2)(m-3)}{1 \;.\; 2 \;.\; 3 \;.\; 4}.$$

&c., &c., &c.

Substituting these values in equation (2), we obtain

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1 \;.\; 2}z^2 + \frac{m(m-1)(m-2)}{1 \;.\; 2 \;.\; 3}z^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1 \;.\; 2 \;.\; 3 \;.\; 4}z^4 + \text{\&c.} \quad \text{- -} \quad (9).$$

If now, in the last equation, we write $\frac{a}{x}$ for z, and then multiply both members by x^m, we shall have,

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1 \;.\; 2}a^2x^{m-2} + \frac{m(m-1)(m-2)}{1 \;.\; 2 \;.\; 3}a^3x^{m-3}$$
$$+ \text{\&c.} \;.\;.\; (10).$$

Hence, we conclude, since this formula is identical with that deduced in Art. 136, that the form of the development of $(x+a)^m$ will be the same, whether m is *positive or negative, entire or fractional.*

It is plain that the number of terms of the development, when m is either fractional or negative, will be infinite.

Applications of the Binomial Formula.

204. If in the formula $(x+a)^m =$

$$x^m\left(1 + m\,.\,\frac{a}{x} + m\,.\,\frac{m-1}{2}\,.\,\frac{a^2}{x^2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{a^3}{x^3} + \;.\;.\;.\;\right)$$

we make $m = \frac{1}{n}$, it becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}\cdot\frac{a}{x} + \frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{a^2}{x^2} + \frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}\cdot\frac{a^3}{x^3} + \ldots\right)$$

or, reducing, $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}\cdot\frac{a}{x} - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2} + \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3} - \ldots\right)$$

The fifth term, within the parenthesis, can be found by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, then changing the sign of the result, and so on.

205. The formula just deduced may be used to find an approximate root of a number. Let it be required to find, by means of it, the cube root of 31.

The greatest perfect cube in 31 is 27. Let $x = 27$ and $a = 4$: making these substitutions in the formula, and putting 3 in the place of n, it becomes

$$\sqrt[3]{31} = 3\left(1 + \frac{1}{3}\cdot\frac{4}{27}\cdot - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683}\right.$$
$$\left. - \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{2}{3}\cdot\frac{256}{531441} + \text{\&c.}\right)$$

or, by reducing,

$$\sqrt[3]{31} = 3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \frac{2560}{43046721} + \text{\&c.}$$

Whence, $\sqrt[3]{31} = 3\,.\,14138$, which, as we shall show presently, is exact to within less than .00001.

We may, in like manner, treat all similar cases: hence, for extracting any root, approximatively, by the binomial formula, we have the following

RULE.

Find the perfect power of the degree indicated, which is nearest to the given number, and place this in the formula for x. Subtract this power from the given number, and substitute this difference, which will often be negative, in the formula for a. Perform the operations indicated, and the result will be the required root

EXAMPLES.

1. $\sqrt[3]{28} = 27^{\frac{1}{3}}\left(1 + \frac{1}{27}\right)^{\frac{1}{3}} = 3.0366.$

2. $\sqrt[5]{30} = (32 - 2)^{\frac{1}{5}} = 32^{\frac{1}{5}}\left(1 - \frac{1}{16}\right)^{\frac{1}{5}} = 1.9744.$

3. $\sqrt[5]{39} = (32 + 7)^{\frac{1}{5}} = 32^{\frac{1}{5}}\left(1 + \frac{7}{32}\right)^{\frac{1}{5}} = 2.0807.$

4. $\sqrt[7]{108} = (128 - 20)^{\frac{1}{7}} = 128^{\frac{1}{7}}\left(1 - \frac{5}{32}\right)^{\frac{1}{7}} = 1.95204.$

206. When the terms of a series go on decreasing in value, the series is called a *decreasing* series; and when they go on increasing in value, it is called an *increasing* series.

A *converging series* is one in which the greater the number of terms taken, the nearer will their sum approximate to a fixed value, which is the *true sum of the series.* When the terms of a decreasing and converging series are *alternately positive and negative*, as in the first example above, we can determine the degree of approximation when we take the sum of a limited number of terms for the true sum of the series.

For, let $a - b + c - d + e - f + \ldots$, &c., be a decreasing series, $b, c, d, \ldots$ being positive quantities, and let x denote the true sum of this series. Then, if n denote the number of terms taken, the value of x will be found between the sums of n and $n + 1$ terms.

For, take any two consecutive sums,

$$a - b + c - d + e - f, \quad \text{and} \quad a - b + c - d + e - f + g.$$

In the first, the terms which follow $-f$, are

$$+ g - h, \quad + k - l + \ldots;$$

but, since the series is decreasing, the terms $g - h$, $k - l \ldots$ &c., are positive; therefore, in order to obtain the complete value of x, a positive number must be added to the sum $a - b + c - d + e - f$. Hence, we have

$$a - b + c - d + e - f < x.$$

In the second sum, the terms which follow $+g$, are $-h + k - l + m \ldots$ Now, $-h+k$, $-l+m \ldots$ &c., are negative; therefore, in order to obtain the sum of the series, a negative quantity must be added to

$$a - b + c - d + e - f + g;$$

or, in other words, it is necessary to diminish it. Consequently,

$$a - b + c - d + e - f + g > x.$$

Therefore, *x is comprehended between the sums of the first n and the first $n+1$ terms.*

But the difference between these two sums is equal to g; and since x is comprised between them, g must be greater than the difference between x and either of them; hence, the *error committed by taking the sum of n terms, $a - b + c - d + e - f$, of the series, for the sum of the series is numerically less than the following term.*

207. The binomial formula serves also to develop algebraic expressions into series.

EXAMPLES.

1. To develop the expression $\dfrac{1}{1-z}$, we have,

$$\frac{1}{1-z} = (1-z)^{-1}.$$

In the binomial formula, make $m = -1$, $x = 1$, and $a = -z$, and it becomes

$$(1-z)^{-1} = 1 - 1\,.\,(-z) - 1\,.\,\frac{-1-1}{2}\,.\,(-z)^2$$
$$-1\,.\,\frac{-1-1}{2}\,.\,\frac{-1-2}{3}\,.\,(-z)^3 - \ldots$$

or, performing the operations indicated, we find for the development,

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \text{\&c.}$$

We might have obtained this result, by applying the rule for division.

2. Again take the expression,

$$\frac{2}{(1-z)^3} \quad \text{or} \quad 2(1-z)^{-3}.$$

Substituting in the binomial formula -3 for m, 1 for x, and $-z$ for a, it becomes,

$$(1-z)^{-3} = 1 - 3.(-z) - 3.\frac{-3-1}{2}.(-z)^2$$
$$-3.\frac{-3-1}{2}.\frac{-3-2}{3}.(-z)^3 - \&c.$$

Performing the indicated operations and multiplying by 2, we find

$$\frac{2}{(1-z)^3} = 2(1 + 3z + 6z^2 + 10z^3 + 15z^4 + \&c.).$$

3. To develop the expression $\sqrt[3]{2z - z^2}$ we first place it under the form $\sqrt[3]{2z} \times \left(1 - \frac{z}{2}\right)^{\frac{1}{3}}$. By the application of the binomial formula, we find

$$\left(1 - \frac{z}{2}\right)^{\frac{1}{3}} = 1 + \frac{1}{3}\left(-\frac{z}{2}\right) + \frac{1}{3}.\frac{\frac{1}{3}-1}{2}.\left(-\frac{z}{2}\right)^2 + \ldots$$
$$= 1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 - \ldots;$$

hence, $\sqrt[3]{2z - z^2} = \sqrt[3]{2z}\left(1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 -, \&c.\right)$

4. Develop the expression $\frac{1}{(a+b)^2} = (a+b)^{-2}$ into a series

5. Develop $\frac{r^2}{r+x}$ into a series.

Ans. $r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3}$, &c.

6. Develop the square root of $\frac{a^2 + x^2}{a^2 - x^2}$ into a series.

Ans. $1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^6}$, &c.

7. Develop the cube root of $\frac{a^2}{(a^2 + x^2)^2}$ into a series.

Ans. $\frac{1}{a^{\frac{2}{3}}} \times \left(1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}, \&c.\right)$

Summation of Series.

208. The *Summation of a Series*, is the operation of finding an expression for the sum of any number of terms. Many useful series may be *summed* by the aid of two auxiliary series.

Let there be a *given* series, whose terms may be derived from the expression $\frac{q}{n(n+p)}$, by giving to p a fixed value, and then attributing suitable values to q and n.

Let there be two auxiliary series formed from the expressions $\frac{q}{n}$ and $\frac{q}{n+p}$, so that the values of p, q, and n, shall be the same as in the corresponding terms of the first series.

It can easily be shown that any term of the first series is equal to $\frac{1}{p}$ multiplied by the excess of the corresponding term in the second series, over that in the third.

For, if we take the expression

$$\frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right),$$

and perform the operations indicated, we shall get the expression,

$$\frac{q}{n(n+p)}; \quad \text{hence, we have}$$

$$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right);$$

which was to be proved.

It follows, therefore, that *the sum of any number of terms of the first series, is equal to* $\frac{1}{p}$ *multiplied by the excess of the sum of the corresponding terms in the second series, over that of the corresponding terms in the third series.*

Whenever, therefore, we can find this last difference, it is always possible to sum the given series.

EXAMPLES.

1. Required the sum of n terms of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\text{\&c.}$$

Comparing the terms of this series with the expression

$$\frac{q}{n(n+p)},$$

we see that making $p=1$, $q=1$, and $n=1, 2, 3, 4$, &c., in succession, will produce the given series.

The two corresponding auxiliary series, to n terms, are

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\frac{1}{n},$$

and
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\frac{1}{n}+\frac{1}{n+1}.$$

The difference between the sums of n terms of the first and second auxiliary series is

$$1-\frac{1}{n+1},$$ or, if we denote the sum of n terms of the given series by S, we have,

$$S=1-\frac{1}{n+1}.$$

If the number of terms is infinite $n=\infty$ and

$$S=1.$$

2. Required the sum of n terms of the series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\text{\&c.},$$

If we compare the terms of this series with the expression

$$\frac{q}{n(n+p)},$$

we see that $p=2$, $q=1$, and $n=1, 3, 5, 7$, &c., in succession.

The two auxiliary series, to n terms, are,

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2n-1},$$

and $$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n+1};$$

hence, as before,

$$S=\frac{1}{2}\left(1-\frac{1}{2n+1}\right).$$

If $n=\infty$, we find $S=\frac{1}{2}$.

3. Required the sum of n terms of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\text{\&c.}$$

Here $p=3,\quad q=1,\quad n=1,\ 2,\ 3,\ 4,$ &c.

The two auxiliary series, to n terms, are,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n},$$

$$\frac{1}{4}+\frac{1}{5}\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3};$$

hence, $$S=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right).$$

If $$n=\infty,\ S=\frac{11}{18}.$$

4. Required the sum of the series

$$\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}+\frac{4}{17.21}+\text{\&c.}$$

Ans. 1.

5. Find the sum of n terms of the series,

$$\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\frac{6}{11.13}-\text{\&c.}\ldots$$

Here $p=2,\ q=2,\ -3,\ +4,\ -5,\ +6,$ &c.

$n=3,\ 5,\ 7,\ 9,\ 11,$ &c.

The two auxiliary series are,

$$\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \ldots\ldots \mp \frac{n+1}{2n+1}$$

$$+ \frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \ldots\ldots \pm \frac{n}{2n+1} \mp \frac{n+1}{2n+3};$$

hence, $S = \frac{1}{2}\left(\frac{2}{3} \pm \frac{n+1}{2n+3}\right) - \frac{1}{2}(1 - 1 + 1 - \ldots \pm 1).$

If n is *even*, the upper sign is used, and the quantity in the last parenthesis becomes $+1$, in which case

$$S = \frac{1}{2}\left(\frac{2}{3} + \frac{n+1}{2n+3}\right) - \frac{1}{2} = \frac{1}{2}\left(-\frac{1}{3} + \frac{n+1}{2n+3}\right).$$

If n is *odd*, the lower sign is used, and the quantity in the last parenthesis becomes 0, in which case

$$S = \frac{1}{2}\left(\frac{2}{3} - \frac{n+1}{2n+3}\right).$$

If in either formula we make

$$n = \infty, \quad \frac{n+1}{2n+3} = \frac{1 + \frac{1}{n}}{2 + \frac{3}{n}} \text{ becomes } \frac{1}{2}, \text{ and } S = \frac{1}{12}.$$

6. Find the sum of n terms of the series,

$$\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \frac{1}{4.6}, \text{ \&c.}$$

Here, $p = 2,\ q = 1,\ -1,\ +1,\ -1, +1,\ -1$, &c.

$$n = 1,\ 2,\ 3,\ 4, \text{ \&c.}$$

The two auxiliary series are,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots \mp \frac{1}{n}$$

$$+ \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots \mp \frac{1}{n} \pm \frac{1}{n+1} \mp \frac{1}{n+2}$$

whence, $S = \frac{1}{2}\left(\frac{1}{2} \mp \frac{1}{n+1} \pm \frac{1}{n+2}\right).$

If $n = \infty$, we find $S = \frac{1}{4}.$

Of the Method by Differences.

209. Let $a, b, c, d \ldots$ &c., represent the successive terms of a series formed according to any fixed law; then if each term be subtracted from the succeeding one, the several remainders will form a new series called the *first order of differences.* If we subtract each term of this series from the succeeding one, we shall form another series called the *second order of differences,* and so on, as exhibited in the annexed table.

$$\begin{array}{llllll}
a, \ b, & c, & d, & e, & \\
\ \ b-a, & c-b, & d-c, & e-d, \text{\&c.,} & \text{1st.} \\
& c-2b+a, & d-2c+b, & e-2d+c, \text{\&c.,} & \text{2d.} \\
& & d-3c+3b-a, & e-3d+3c-b, \text{\&c.,} & \text{3d.} \\
& & & e-4d+6c-4b+a, \text{\&c.,} & \text{4th.}
\end{array}$$

If, now, we designate the first terms of the first, second, third, &c. orders of differences, by d_1, d_2, d_3, d_4, &c., we shall have,

$$\begin{array}{ll}
d_1 = b - a, & \text{whence } b = a + d_1, \\
d_2 = c - 2b + a, & \text{whence } c = a + 2d_1 + d_2, \\
d_3 = d - 3c + 3b - a, & \text{whence } d = a + 3d_1 + 3d_2 + d_3, \\
d_4 = e - 4d + 6c - 4b + a, & \text{whence } e = a + 4d_1 + 6d_2 + 4d_3 + d_4, \\
\quad \text{\&c.} \quad \text{\&c.} & \quad \text{\&c.} \quad \text{\&c.}
\end{array}$$

And if we designate the term of the series which has n terms before it, by T, we shall find, by a continuation of the above process,

$$T = a + nd_1 + \frac{n.(n-1)}{1 \ . \ 2} d_2 + \frac{n.(n-1)(n-2)}{1 \ . \ 2 \ . \ 3} d_3$$

$$+ \frac{n.(n-1)(n-2)(n-3)}{1 \ . \ 2 \ . \ 3 \ . \ 4} d_4 + \text{\&c.} \quad - \ - \ - \quad (1).$$

This formula enables us to find the $(n+1)^{th}$ term of a series when we know the first terms of the successive orders of differences.

210. To find an expression for the sum of n terms of the series a, b, c, &c., let us take the series

$$0,\ a,\ a+b,\quad a+b+c,\quad a+b+c+d,\ \&c.\ \ -\ -\ -\ -\ .\ (2)$$

The first order of differences is evidently

$$a,\ b,\ c,\ d,\ \ldots\ldots,\ \&c.\ \ -\ -\ -\ -\ -\ -\ -\ .\ (3)$$

Now, it is obvious that the sum of n terms of the series (3), is equal to the $(n+1)^{th}$ term of the series (2).

But the first term of the first order of differences in series (2) is a; the first term of the second order of differences is the same as d_1 in equation (1). The first term of the third order of differences is equal to d_2, and so on.

Hence, making these changes in formula (1), and denoting the sum of n terms by S, we have,

$$S = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3$$
$$+\ \&c.\ \ -\ -\ -\ -\ (4).$$

When all of the terms of any order of differences become equal, the terms of all succeeding orders of differences are 0, and formulas (1) and (4) give exact results. When there are no orders of differences, whose terms become equal, then formulas do not give exact results, but approximations more or less exact according to the number of terms used.

EXAMPLES.

1. Find the sum of n terms of the series 1.2, 2.3, 3.4, 4.5, &c.

Series,	1.2,		2.3,		3.4,		4.5.		5.6, &c.
1st order of differences,		4,		6,		8,		10, &c.	
2d order of differences,			2,		2,		2, &c.		
3d order of differences,				0,		0.			

Hence, we have, $a = 2$, $d_1 = 4$, $d_2 = 2$, d_3, d_4, &c., equal to 0.

Substituting these values for a, d_1, d_2, &c., in formula (4), we find,

$$S = 2n + \frac{n(n-1)}{1.2} \times 4 + \frac{n(n-1)(n-2)}{1.2.3} \times 2;$$

whence, $$S = \frac{n(n+1)(n+2)}{3}.$$

2. Find the sum of n terms of the series 1.2.3, 2.3.4, 3.4.5, 4.5.6, &c.

1st order of differences,	18,	36,	60,	90,	126, &c.
2d order of differences,		18,	24,	30,	36, &c.
3d order of differences,			6,	6,	6, &c.
4th order of differences,				0,	0. &c.

We find $a = 6$, $d_1 = 18$, $d_2 = 18$, $d_3 = 6$, $d_4 = 0$, &c.

Substituting in equation (4), and reducing, we find,

$$S = \frac{n(n+1)(n+2)(n+3)}{4}.$$

3. Find the sum of n terms of the series 1, $1+2$, $1+2+3$, $1+2+3+4$, &c.

Series,	1,	3,	6,	10,	15,	21.
1st order of differences,	2,	3,	4,	5,	6.	
2d order of differences,	1,	1,	1,	1.		
3d order of differences,	0,	0,	0,			

$$a = 1, \quad d_1 = 2, \quad d_2 = 1, \quad d_3 = 0, \quad d_4 = 0, \text{ \&c.};$$

hence, $$S = n + \frac{n(n-1)}{1.2}.2 + \frac{n(n-1)(n-2)}{1.2.3} = \frac{n^3 + 3n^2 + 2n}{1.2.3};$$

or, reducing, $$S = \frac{n(n+1)(n+2)}{1.2.3}.$$

4. Find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, &c.

We find, $a = 1$, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, $d_4 = 0$, &c., &c.

Substituting these values in formula (4), and reducing, we find,

$$S = \frac{n(n+1)(2n+1)}{1.2.3}.$$

5. Find the sum of n terms of the series,

$$1.(m+1),\quad 2(m+2),\quad 3(m+3),\quad 4(m+4),\ \&c.$$

We find, $a=m+1$, $d_1=m+3$, $d_2=2$, $d_3=0$, &c.;

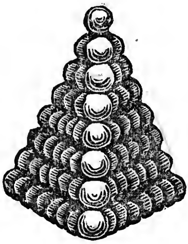
whence, $S=n(m+1)+\dfrac{n.(n-1)}{1\,.\,2}(m+3)+\dfrac{n.(n-1)(n-2)}{1\,.\,2\,.\,3}\times 2,$

or, $$S=\frac{n.(n+1).(1+2n+3m)}{1\,.\,2\,.\,3}$$

Of Piling Balls.

The last three formulas deduced, are of practical application in determining the number of balls in different shaped piles.

First, in the Triangular Pile.

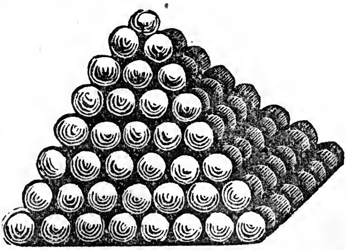


211. A triangular pile is formed of successive triangular layers, such that the number of shot in each side of the layers, decreases continuously by 1 to the single shot at the top. The number of balls in a complete triangular pile is evidently equal to the sum of the series 1, $1+2$, $1+2+3$, $1+2+3+4$, &c. to $1+2+\ldots+n$, n denoting the number of balls on one side of the base.

But from example 3d, last article, we find the sum of n terms of the series,

$$S=\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\quad\cdot\ \cdot\quad(1).$$

Second, in the Square Pile.

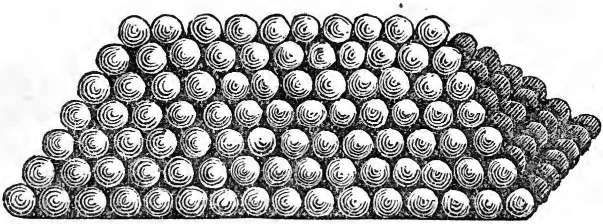


212. The square pile is formed, as shown in the figure. The number of balls in the top layer is 1; the number in the second layer is denoted by 2^2; in the next, by 3^2, and so on. Hence, the number of balls in a pile of n layers, is equal to the sum of the series, 1^2, 2^2 3^2,

&c., n^2, which we see, from example 4th of the last article, is

$$S = \frac{n.(n+1).(2n+1)}{1 \; . \; 2 \; . \; 3} \quad - \; - \; - \; - \quad (2).$$

Third, in the Oblong Pile.



213. The complete oblong pile has $(m+1)$ balls in the upper layer, $2.(m+2)$ in the next layer, $3(m+3)$ in the third, and so on: hence, the number of balls in the complete pile, is given by the formula deduced in example 5th of the preceding article,

$$S = \frac{n.(n+1).(1+2n+3m)}{1 \; . \; 2 \; . \; 3} \quad - \; - \; - \; - \quad (3).$$

214. If any of these piles is incomplete, compute the number of balls that it would contain if complete, and the number that would be required to complete it; the excess of the former over the latter, will be the number of balls in the pile.

The formulas (1), (2) and (3) may be written,

triangular, $$S = \frac{1}{3} . \frac{n(n+1)}{2}(n+1+1) \quad - \; - \; - \; - \; - \; - \quad (1);$$

square, $$S = \frac{1}{3} . \frac{n(n+1)}{2}(n+n+1) \quad - \; - \; - \; - \; - \; - \quad (2);$$

rectangular, $$S = \frac{1}{3} . \frac{n(n+1)}{2}\Big((n+m)+(n+m)+(m+1)\Big) - (3).$$

Now, since $\frac{n(n+1)}{2}$ is the number of balls in the triangular face of each pile, and the next factor, the number of balls in the longest line of the base, plus the number in the side of the base opposite, plus the parallel top row, we have the following

RULE.

Add to the number of balls in the longest line of the base the number in the parallel side opposite, and also the number in the top parallel row; then multiply this sum by one-third the number in the triangular face; the product will be the number of balls in the pile.

EXAMPLES.

1. How many balls in a triangular pile of 15 courses?

Ans. 680.

2. How many balls in a square pile of 14 courses? and how many will remain after 5 courses are removed?

Ans. 1015 and 960.

3. In an oblong pile, the length and breadth at bottom are respectively 60 and 30: how many balls does it contain?

Ans. 23405.

4. In an incomplete oblong pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top 35 and 9: how many balls does it contain?

Ans. 7190.

5. How many balls in an incomplete triangular pile, the number of balls in each side of the lower course being 20, and in each side of the upper, 10?

6. How many balls in an incomplete square pile, the number in each side of the lower course being 15, and in each side of the upper course 6?

7. How many balls in an incomplete oblong pile, the numbers in the lower courses being 92 and 40; and the numbers in the corresponding top courses being 70 and 18?

CHAPTER IX.

CONTINUED FRACTIONS—EXPONENTIAL QUANTITIES—LOGARITHMS, AND FORMULAS FOR INTEREST.

215. Every expression of the form

$$\cfrac{1}{a+\cfrac{1}{b}}, \qquad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}, \qquad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}$$

in which a, b; c, d, &c., are positive whole numbers, is called a *continued fraction:* hence,

A CONTINUED FRACTION *has* 1 *for its numerator, and for its denominator, a whole number plus a fraction, which has* 1 *for its numerator and for its denominator a whole number plus a fraction, and so on.*

216. The resolution of equations of the form

$$a^x = b,$$

gives rise to continued fractions.

Suppose, for example, $a = 8$, $b = 32$. We then have

$$8^x = 32,$$

in which $x > 1$ and less than 2. Make

$$x = 1 + \frac{1}{y},$$

in which $y > 1$, and the proposed equation becomes

$$32 = 8^{1+\frac{1}{y}} = 8 \times 8^{\frac{1}{y}}; \quad \text{whence,}$$

$$8^{\frac{1}{y}} = 4, \quad \text{and consequently,} \quad 8 = 4^{y}.$$

It is plain, that the value of y lies between 1 and 2. Suppose

$$y = 1 + \frac{1}{z},$$

and we have, $$8 = 4^{1+\frac{1}{z}} = 4 \times 4^{\frac{1}{z}};$$

hence, $$4^{\frac{1}{z}} = 2, \quad \text{and} \quad 4 = 2^{z}, \quad \text{or} \quad z = 2.$$

But, $$y = 1 + \frac{1}{z} = 1 + \frac{1}{2} = \frac{3}{2};$$

and $$x = 1 + \frac{1}{y} = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = 1 + \frac{2}{3} = \frac{5}{3};$$

and this value will satisfy the proposed equation.

For, $$8^{\frac{5}{3}} = \sqrt[3]{8^5} = \sqrt[3]{(2^3)^5} = \sqrt[3]{(2^5)^3} = 2^{5} = 32.$$

217. If we apply a similar process to the equation

$$(10)^x = 200,$$

we shall find

$$x = 2 + \frac{1}{y}; \quad y = 3 + \frac{1}{z}; \quad z = 3 + \frac{1}{u}.$$

Since 200 is not an exact power, x cannot be exactly expressed either by a whole number or a fraction: hence, the value of x will be incommensurable with 1, and the continued fraction will not terminate, but will be of the form

$$x = 2 + \frac{1}{y} = 2 + \cfrac{1}{3 + \cfrac{1}{z}} = 2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{u + \text{\&c.}}}}$$

218. Vulgar fractions may also be placed under the form of continued fractions.

Let us take, for example, the fraction $\frac{65}{149}$, and divide both its terms by the numerator 65, the value of the fraction will not be changed, and we shall have

$$\frac{65}{149} = \frac{1}{\frac{149}{65}},$$

or effecting the division, $$\frac{65}{149} = \frac{1}{2 + \frac{19}{65}}.$$

Now, if we neglect the fractional part, $\frac{19}{65}$, of the denominator, we shall obtain $\frac{1}{2}$ for an approximate value of the given fraction. But this value will be too large, since the *denominator* used is too *small.*

If, on the contrary, instead of neglecting the part $\frac{19}{65}$, we were to replace it by 1, the approximate value would be $\frac{1}{3}$, which would be too small, since the denominator 3 is too *large.* Hence,

$$\frac{1}{2} > \frac{65}{149} \quad \text{and} \quad \frac{1}{3} < \frac{65}{149};$$

therefore the value of the fraction is comprised between $\frac{1}{2}$ and $\frac{1}{3}$.

If we wish a nearer approximation, it is only necessary to operate on the fraction $\frac{19}{65}$ as we did on the given fraction $\frac{65}{149}$, and we obtain,

$$\frac{19}{65} = \frac{1}{3 + \frac{8}{19}};$$

hence, $$\frac{65}{149} = \frac{1}{2 + \frac{1}{3 + \frac{8}{19}}}.$$

If, now, we neglect the part $\frac{8}{19}$, the denominator 3 will be less than the true denominator, and $\frac{1}{3}$ will be *larger* than the number which ought to be added to 2; hence, 1 divided by $2+\frac{1}{3}$ will be *less* than the true value of the fraction; that is, if we stop after the first reduction and omit the last fraction, the result will be too great; if at the second, it will be too small, &c.; and, generally,

If we stop at an odd reduction, and neglect the fractional part that comes after, the result will be too great; but if we stop at an even reduction, and neglect the fractional part that follows, the result will be too small.

219. The separate fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., which make up a continued fraction, are called *integral fractions.*

The fractions,

$$\frac{1}{a}, \quad \frac{1}{a+\frac{1}{b}}, \quad \frac{1}{a+\frac{1}{b+\frac{1}{c}}}, \text{ \&c.}$$

are called *approximating fractions*, because each gives, in succession, a nearer approximation to the true value of the fraction: hence,

An approximating fraction is the result obtained by stopping at any integral fraction, and neglecting all that come after.

If we stop at the first integral fraction, the resulting approximating fraction is said to be of the first order; if at the second integral fraction, the resulting approximating fraction is of the second order, and so on.

When there is a finite number of integral fractions, we shall get the true value of the expression by considering them all: when their number is infinite, only an approximate value can be found.

220. We will now explain the manner in which any approximating fraction may be found from those which precede it.

$$(1) \quad \frac{1}{a} \quad = \frac{1}{a} \qquad \text{1st app. fraction.}$$

$$(2) \quad \frac{1}{a+\dfrac{1}{b}} \quad = \frac{b}{ab+1} \qquad \text{2d app. fraction.}$$

$$(3) \quad \frac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}} \quad = \frac{bc+1}{(ab+1)c+a} \qquad \text{3d app. fraction.}$$

By examining the third approximating fraction, we see that its numerator is formed by multiplying the numerator of the preceding approximating fraction by the denominator of the third integral fraction, and adding to the product the numerator of the first approximating fraction: and that the denominator is formed by multiplying the denominator of the preceding approximating fraction by the denominator of the third integral fraction, and adding to the product the denominator of the first approximating fraction.

Let us now assume that the $(n-1)^{th}$ approximating fraction is formed from the two preceding approximating fractions by the same law, and let $\frac{P}{P'}$, $\frac{Q}{Q'}$, and $\frac{R}{R'}$, designate, respectively, the $(n-3)$, $(n-2)$, and $(n-1)$, approximating fractions.

Then, if m denote the denominator of the $(n-1)^{th}$ integral fraction, we shall have from the assumed law of formation,

$$\frac{R}{R'} = \frac{Qm+P}{Q'm+P'}, \quad \text{- - - - - -} \quad (1).$$

Let us now consider another integral fraction $\frac{1}{n}$, and suppose $\frac{S}{S'}$ to represent the n^{th} approximating fraction. It is plain that we shall obtain the value of $\frac{S}{S'}$, from that of $\frac{R}{R'}$, by simply changing $\frac{1}{m}$ into $\dfrac{1}{m+\dfrac{1}{n}}$, or, by substituting $m+\frac{1}{n}$ for m, in equation (1);

whence, $$\frac{S}{S'} = \frac{Q\left(m + \frac{1}{n}\right) + P}{Q'\left(m + \frac{1}{n}\right) + P'} = \frac{(Q\,m + P\,)n + Q}{(Q'm + P')n + Q'} = \frac{R\,n + Q}{R'n + Q'}.$$

Hence, if the law assumed for the formation of the $(n-1)^{th}$ approximating fraction is true, the same law is true for the formation of the n^{th} approximating fraction. But we have shown that the law is true for the formation of the third; hence, it must be true for the formation of the fourth; being true for the fourth, it is true for the fifth, and so on; hence, it is general. Therefore,

The numerator of the n^{th} approximating fraction is formed by multiplying the numerator of the preceding fraction by the denominator of the n^{th} integral fraction, and adding to the product the numerator of the $(n-2)^{th}$ approximating fraction; and the denominator is formed according to the same law, from the two preceding denominators.

221. If we take the difference between the first and second approximating fractions, we find,

$$\frac{1}{a} - \frac{b}{ab + 1} = \frac{ab + 1 - ab}{a(ab + 1)} = \frac{+1}{a(ab + 1)},$$

and the difference between the second and third is,

$$\frac{b}{ab + 1} - \frac{bc + 1}{(ab + 1)c + a} = \frac{-1}{(ab + 1)\,[(ab + 1)c + a]}.$$

In both these cases we see that the difference between two consecutive approximating fractions is numerically equal to 1, divided by the product of the denominators of the two fractions

To show that this law is general, let

$$\frac{P}{P'}, \qquad \frac{Q}{Q'}, \qquad \frac{R}{R'},$$

be any three consecutive approximating fractions. Then

$$\frac{P}{P'} - \frac{Q}{Q'} = \frac{PQ' - P'Q}{P'Q'}$$

and $$\frac{Q}{Q'} - \frac{R}{R'} = \frac{R'Q - RQ'}{Q'R'}.$$

But $R = Qm + P$, and $R' = Q'm + P'$ (Art. 220).

Substituting these values in the last equation, we have,

$$\frac{Q}{Q'} - \frac{R}{R'} = \frac{(Q'm + P')Q - (Qm + P)Q'}{R'Q'};$$

or, reducing,

$$\frac{Q}{Q'} - \frac{R}{R'} = \frac{P'Q - PQ'}{R'Q'}.$$

Now, if $(PQ' - P'Q)$ is equal to ± 1, then $(P'Q - PQ')$ must be equal to ∓ 1; that is,

If the difference between the $(n-2)$ and the $(n-1)$ fractions, is formed by the assumed law, then the difference between the $(n-1)^{th}$ and the n^{th} fractions must be formed by the same law.

But we have shown that the law holds true for the difference between the second and third fractions; hence, it must be true for the difference between the third and fourth; being true for the difference between the third and fourth, it must be true for the difference between the fourth and fifth, and so on; hence, it is general: that is,

The difference between any two consecutive approximating fractions, is equal to ± 1, divided by the product of their denominators.

When an approximating fraction of an even order is taken from one of an odd order, the upper sign is used: when one of an odd order is taken from one of an even order, the lower sign is used.

This ought to be the case, since we have shown that every approximating fraction of an odd order is greater than the true value of the continued fraction, whilst every one of an even order is less.

222. It has already been shown (Art. 218), that each of the approximating fractions of an odd order, exceeds the true value of the continued fraction; while each one of an even order is less than it. Hence, the difference between any two consecutive approximating fractions is greater than the difference

between either of them and the true value of the continued fraction. Therefore, stopping at the n^{th} approximating fraction, the result will be the true value of the fraction, to within less than 1 divided by the denominator of that fraction, multiplied by the denominator of the approximating fraction which follows.

Thus, if Q' and R' are the denominators of consecutive approximating fractions, and we stop at the fraction whose denominator is Q', the result will be true to within less than $\frac{1}{Q'R'}$. But, since a, b, c, d, &c., are entire numbers, the denominator R' will be greater than Q', and we shall have

$$\frac{1}{Q'R'} < \frac{1}{Q'^2};$$

hence, if the result be true to within less than $\frac{1}{Q'R'}$, it will certainly be true to within less than the larger quantity

$$\frac{1}{Q'^2}; \quad \text{that is,}$$

The approximate result which is obtained, is true to within less than 1 *divided by the square of the denominator of the last approximating fraction that is employed.*

223. If we take the fraction $\frac{829}{347}$, we have,

$$\frac{829}{347} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{19}}}}}.$$

Here, we have in the quotient the whole number 2, which may either be set aside, and added to the fractional part after its value shall have been found, or we may place 1 under it for a denominator, and treat it as an approximating fraction.

Solution of the Equation $a^x = b$.

224. An equation of the form,

$$a^x = b,$$

is called an *exponential equation*. The object in solving this equation is, to find the exponent of the power to which it is necessary to raise a given number a, in order to produce another given number b.

225. Suppose it were required to solve the equation,

$$2^x = 64.$$

By raising 2 to its different powers, we find that

$$2^6 = 64; \quad \text{hence,} \quad x = 6$$

will satisfy the equation.

Again, let there be the equation,

$$3^x = 243, \quad \text{in which} \quad x = 5.$$

Now, so long as the second member b is a *perfect power* of the given number a, the value of x may be obtained by trial, by raising a to its successive powers, commencing at the first, for the exponent of the power will be the value of x.

226. Suppose it were required to solve the equation,

$$2^x = 6.$$

By making $x = 2$, and $x = 3$, we find,

$$2^2 = 4 \quad \text{and} \quad 2^3 = 8;$$

from which we perceive that the value of x is comprised between 2 and 3.

Make, then, $\quad x = 2 + \dfrac{1}{x'}, \quad$ in which $\quad x' > 1.$

Substituting this value in the given equation, it becomes,

$$2^{2+\frac{1}{x'}} = 6, \quad \text{or} \quad 2^2 \times 2^{\frac{1}{x'}} = 6; \quad \text{hence,}$$

$$2^{\frac{1}{x'}} = \frac{6}{4} = \frac{3}{2}.$$

and by changing the order of the members, and raising both to the x' power,

$$\left(\frac{3}{2}\right)^{x'} = 2.$$

To determine x', make x' successively equal to 1 and 2; we find,

$$\left(\frac{3}{2}\right)^1 = \frac{3}{2} < 2; \quad \text{and} \quad \left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2;$$

therefore, x' is comprised between 1 and 2.

Make, $\qquad x' = 1 + \dfrac{1}{x''}$, in which $x'' > 1$.

By substituting this value in the equation $\left(\frac{3}{2}\right)^{x'} = 2$,

we find, $\qquad \left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2; \quad$ hence, $\quad \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}} = 2,$

and consequently, 4/3 $\qquad \left(\frac{4}{3}\right)^{x''} = \frac{3}{2}.$

The supposition, $\qquad x'' = 1, \quad$ gives $\quad \frac{4}{3} < \frac{3}{2};$

and $\qquad x'' = 2, \quad$ gives $\quad \frac{16}{9} > \frac{3}{2};$

therefore, x'' is comprised between 1 and 2.

Let $\qquad x'' = 1 + \dfrac{1}{x'''}; \quad$ then,

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2}; \quad \text{hence,} \quad \frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2},$$

whence, $\qquad \left(\frac{9}{8}\right)^{x'''} = \frac{4}{3}.$

If we make $x''' = 2$, we have

$$\left(\frac{9}{8}\right)^2 = \frac{81}{64} < \frac{4}{3},$$

and if we make $x''' = 3$, we have

$$\left(\frac{9}{8}\right)^3 = \frac{729}{512} > \frac{4}{3};$$

therefore, x''' is comprised between 2 and 3.

Make $$x''' = 2 + \frac{1}{x^{\text{IV}}}, \text{ and we have}$$

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{\text{IV}}}} = \frac{4}{3}; \quad \text{hence,} \quad \frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{x^{\text{IV}}}} = \frac{4}{3};$$

and consequently, $$\left(\frac{256}{243}\right)^{x^{\text{IV}}} = \frac{9}{8}.$$

Operating upon this exponential equation in the same manner as upon the preceding equations, we shall find two entire numbers, 2 and 3, between which x^{IV} will be comprised.

Making

$$x^{\text{IV}} = 2 + \frac{1}{x^{\text{V}}},$$

x^{V} can be determined in the same manner as x^{IV}, and so on.

Making the necessary substitutions in the equations

$$x = 2 + \frac{1}{x'}, \quad x' = 1 + \frac{1}{x''}, \quad x'' = 1 + \frac{1}{x'''}, \quad x''' = 2 + \frac{1}{x^{\text{IV}}} \quad \ldots,$$

we obtain the value of x under the form of a whole number, plus a continued fraction.

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{x^{\text{V}}}}}}};$$

hence, we find the first three approximating fractions to be

$$\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{3}{5},$$

and the fourth is equal to

$$\frac{3 \times 2 + 1}{5 \times 2 + 2} = \frac{7}{12} \text{ (Art. 220)},$$

which is the true value of the fractional part of x to within less than

$$\frac{1}{(12)^2}, \quad \text{or} \quad \frac{1}{144} \text{ (Art. 222).}$$

Therefore,

$$x = 2 + \frac{7}{12} = \frac{31}{12} = 2.58333 + \quad \text{to within less than } \frac{1}{144},$$

and if a greater degree of exactness is required, we must take a greater number of integral fractions.

EXAMPLES.

$3^x = 15$	. . . $x =$	2.46	to within less than 0.01.
$(10)^x = 3$	. . . $x =$	0.477	" " 0.001.
$5^x = \frac{2}{3}$	. . . $x =$	$-$ 0.25	" " 0.01.

Of Logarithms.

227. If we suppose a to preserve a constant value in the equation

$$a^x = N,$$

whilst N is made, in succession, equal to every possible number, it is plain that x will undergo changes corresponding to those made in N. By the method explained in the last article, we can determine, for each value of N, the corresponding value of x, either exactly or approximatively.

The value of x, corresponding to any assumed value of the number N, is called the logarithm of that number; and a is called the *base of the system* in which the logarithm is taken. Hence,

The logarithm of a number is the exponent of the power to which it is necessary to raise the base, in order to produce the given number. The logarithms of all numbers corresponding to a given base constitute a system of logarithms.

Any positive number except 1 may be taken as the base of a system of logarithms, and if for that particular base, we suppose the logarithms of all numbers to be computed, they will constitute what is called a *system of logarithms.* Hence, we see that there is an infinite number of systems of logarithms.

228. The base of the *common* system of logarithms is 10, and if we designate the logarithm of any number taken in that system by log, we shall have,

$$
\begin{array}{llll}
(10)^0 = & 1; & \text{whence,} & \log\ 1 = 0; \\
(10)^1 = & 10; & \text{whence,} & \log\ 10 = 1; \\
(10)^2 = & 100; & \text{whence,} & \log\ 100 = 2; \\
(10)^3 = & 1000; & \text{whence,} & \log\ 1000 = 3; \\
& \&c., & & \&c.
\end{array}
$$

We see, that in the common system, the logarithm of any number between 1 and 10, is found between 0 and 1. The logarithm of any number between 10 and 100, is between 1 and 2; the logarithm of any number between 100 and 1000, is between 2 and 3; and so on.

The logarithm of any number, which is not a perfect power of the base, will be equal to a whole number, *plus* a fraction, the value of which is generally expressed decimally. The entire part is called the *characteristic*, and sometimes the *index*.

By examining the several powers of 10, we see, that if a number is expressed by a single figure, the characteristic of its logarithm will be 0; if it is expressed by two figures, the characteristic of its logarithm will be 1; if it is expressed by three figures, the characteristic will be 2; and if it is expressed by n places of figures, the characteristic will be $n - 1$.

If the number is less than 1, its logarithm will be negative, and by considering the powers of 10, which are denoted by negative exponents, we shall have,

$$
\begin{array}{llll}
(10)^{-1} = \dfrac{1}{10} & = .1; & \text{whence,} & \log\ .1 = -1. \\
(10)^{-2} = \dfrac{1}{100} & = .01; & \text{whence,} & \log\ .01 = -2. \\
(10)^{-3} = \dfrac{1}{1000} & = .001; & \text{whence,} & \log\ .001 = -3. \\
\&c., & \&c. & & \&c., \quad \&c.
\end{array}
$$

Here we see that the logarithm of every number between 1 and .1 will be found between 0 and -1; that is, it will be equal to -1, *plus* a fraction less than 1. The logarithm of any number

between .1 and .01 will be between -1 and -2; that is, it will be equal to -2, *plus* a fraction. The logarithm of any number between .01 and .001, will be between -2 and -3, or will be equal to -3, *plus* a fraction, and so on.

In the first case, the characteristic is -1, in the second -2, in the third -3, and in general, *the characteristic of the logarithm of a decimal fraction is negative, and numerically* 1 *greater than the number of* 0's *which immediately follow the decimal point.* The decimal part is always positive, and to indicate that the negative sign extends only to the characteristic, it is generally written over it; thus,

$$\log 0.012 = \bar{2}.079181, \quad \text{which is equivalent to} \quad -2 + .079181.$$

228*. A table of logarithms, is a table containing a set of numbers, and their logarithms so arranged that we may, by its aid, find the logarithm of any number from 1 to a given number, generally 10,000.

The following table shows the logarithms of the numbers, from 1 to 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

When the number exceeds 100, the characteristic of its logarithm is not written in the table, but is always known, since it is 1 less than the number of places of figures of the given number. Thus, in searching for the logarithm of 2970, in a table of logarithms, we should find opposite 2970, the decimal part .472756. But since the number is expressed by four figures, the characteristic of the logarithm is 3. Hence,

$$\log 2970 = 3.472756,$$

and by the definition of a logarithm, the equation

$$a^x = N, \quad \text{gives}$$

$$10^{3.472756} = 2970.$$

General Properties of Logarithms.

229. The general properties of logarithms are entirely independent of the value of the base of the system in which they are taken. In order to deduce these properties, let us resume the equation,

$$a^x = N,$$

in which we may suppose a to have any positive value except 1.

230. If, now, we denote any two numbers by N' and N'', and their logarithms, taken in the system whose base is a, by x' and x'', we shall have, from the definition of a logarithm,

$$a^{x'} = N' \quad - \; - \; - \; - \; - \; - \quad (1),$$

and,

$$a^{x''} = N'' \quad - \; - \; - \; - \; - \; - \quad (2).$$

If we multiply equations (1) and (2) together, member by member, we get,

$$a^{x'+x''} = N' \times N'' \quad - \; - \; - \quad (3).$$

But since a is the base of the system, we have from the definition,

$$x' + x'' = \log(N' \times N''); \quad \text{that is,}$$

The logarithm of the product of two numbers is equal to the sum of their logarithms.

231. If we divide equation (1) by equation (2), member by member, we have,

$$a^{x'-x''} = \frac{N'}{N''} \quad - \ - \ - \ - \ - \ - \quad (4)$$

But, from the definition,

$$x' - x'' = \log\left(\frac{N'}{N''}\right); \quad \text{that is,}$$

The logarithm of the quotient which arises from dividing one number by another is equal to the logarithm of the dividend minus the logarithm of the divisor.

232. If we raise both members of equation (1) to the n^{th} power, we have,

$$a^{nx'} = N'^n \quad - \ - \ - \ - \ - \ - \quad (5).$$

But from the definition, we have,

$$nx' = \log(N'^n); \quad \text{that is,}$$

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

233. If we extract the n^{th} root of both members of equation (1), we shall have,

$$a^{\frac{x'}{n}} = (N')^{\frac{1}{n}} = \sqrt[n]{N'} \quad - \ - \quad (6).$$

But from the definition,

$$\frac{x'}{n} = \log(\sqrt[n]{N'}); \quad \text{that is,}$$

The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

234. From the principles demonstrated in the four preceding articles, we deduce the following practical rules:—

First, To multiply quantities by means of their logarithms.

Find from a table, the logarithms of the given factors, take the sum of these logarithms, and look in the table for the corresponding number; this will be the product required.

Thus,	log 7 - - - - - - -	0.845098
	log 8 - - - - - - -	0.903090
	log 56 - - - - - - - - -	1.748188 ;

hence, $7 \times 8 = 56.$

Second. To divide quantities by means of their logarithms.

Find the logarithm of the dividend and the logarithm of the divisor, from a table; subtract the latter from the former, and look for the number corresponding to this difference; this will be the quotient required.

Thus,	log 84 - - - - - - - - -	1.924279
	log 21 - - - - - - - - -	1.322219
	log 4 - - - - - - - - -	0.602060 ;

hence, $\frac{84}{21} = 4.$

Third, To raise a number to any power.

Find from a table the logarithm of the number, and multiply it by the exponent of the required power; find the number corresponding to this product, and it will be the required power.

Thus,	log 4 - - - - - - - - -	0.602060
		3
	log 64 - - - - - - -	1.806180 ;

hence, $(4)^3 = 64.$

Fourth, To extract any root of a number.

Find from a table the logarithm of the number, and divide this by the index of the root; find the number corresponding to this quotient, and it will be the root required.

Thus,	log 64 - - - - - - - -	1.806180(6
	log 2 - - - - - - -	.301030 ;

hence, $\sqrt[6]{64} = 2.$

By the aid of these principles, we may write the following equivalent expressions :—

$$\text{Log } (a \,.\, b \,.\, c \,.\, d \ldots.) = \log a + \log b + \log c \ldots.$$

$$\text{Log } \left(\frac{abc}{de}\right) = \log a + \log b + \log c - \log d - \log e.$$

$$\text{Log } (a^m \,.\, b^n \,.\, c^p \ldots.) = m \log a + n \log b + p \log c + \ldots.$$

$$\text{Log } (a^2 - x^2) = \log (a + x) + \log (a - x).$$

$$\text{Log } \sqrt{(a^2 - x^2)} = \tfrac{1}{2} \log (a + x) + \tfrac{1}{2} \log (a - x).$$

$$\text{Log } (a^3 \times \sqrt[4]{a^3}) = 3\tfrac{3}{4} \log a.$$

234. We have already explained the method of determining the characteristic of the logarithm of a decimal fraction, in the common system, and by the aid of the principle demonstrated in Art. 231, we can show

That the decimal part of the logarithm is the same as the decimal part of the logarithm of the numerator, regarded as a whole number.

For, let a denote the numerator of the decimal fraction, and let m denote the number of decimal places in the fraction, then will the fraction be equal to

$$\frac{a}{10^m},$$

and its logarithm may be expressed as follows:

$$\log \frac{a}{10^m} = \log a - \log (10)^m = \log a - m \log 10 = \log a - m,$$

but m is a whole number, hence the decimal part of the logarithm of the given fraction is equal to the decimal part of $\log a$, or of the logarithm of the numerator of the given fraction.

Hence, to find the logarithm of a decimal fraction from the common table,

Look for the logarithm of the number, neglecting the decimal point, and then prefix to the decimal part found a negative characteristic equal to 1 *more than the number of zeros which immediately follow the decimal point in the given decimal.*

The rules given for finding the characteristic of the logarithms taken in the common system, will not apply in any other system, nor could we find the logarithm of decimal fractions

directly from the tables in any other system than that whose base is 10.

These are some of the advantages which the common system possesses over every other system.

235. Let us again resume the equation

$$a^x = N.$$

1st. If we make $N = 1$, x must be equal to 0, since $a^0 = 1$; that is,

The logarithm of 1 *in any system is* 0.

2d. If we make $N = a$, x must be equal to 1, since $a^1 = a$: that is,

Whatever be the base of a system, its logarithm, taken in that system, is equal to 1.

Let us, in the equation,

$$a^x = N,$$

First, suppose $a > 1$.

Then, when $N = 1$, $x = 0$; when $N > 1$, $x > 0$; when $N < 1$, $x < 0$, or negative; that is,

In any system whose base is greater than 1, *the logarithms of all numbers greater than* 1 *are positive, those of all numbers less than* 1 *are negative.*

If we consider the case in which $N < 1$, we shall have

$$a^{-x} = N, \quad \text{or} \quad \frac{1}{a^x} = N.$$

Now, if N diminishes, the corresponding values of x must increase, and when N becomes less than any assignable quantity, or 0, the value of x must be ∞: that is,

The logarithm of 0, *in a system whose base is greater than* 1, *is equal to* $-\infty$.

Second, suppose $a < 1$.

Then, when $N = 1$, $x = 0$; when $N < 1$, $x > 0$; when $N > 1$, $x < 0$, or negative: that is,

In any system whose base is less than 1, *the logarithms of all numbers greater than* 1 *are negative, and those of all numbers less than* 1 *are positive.*

If we consider the case in which $N < 1$, we shall have $a^x = N$, in which, if N be diminished, the value of x must be increased; and finally, when $N = 0$, we shall have $x = \infty$: that is,

The logarithm of 0, *in a system whose base is less than* 1, *is equal to* $+\infty$.

Finally, whatever values we give to x, the value of a^x or N will always be positive; whence we conclude that *negative numbers have no logarithms.*

Logarithmic Series.

236. The method of resolving the equation,

$$a^x = b,$$

explained in Art. 226, gives an idea of the construction of logarithmic tables; but this method is laborious when it is necessary to approximate very near the value of x. Analysts have discovered much more expeditious methods for constructing new tables, or for verifying those already calculated. These methods consist in the development of logarithms into series.

If we take the equation,

$$a^x = y,$$

and regard a as the base of a system of logarithms, we shall have,

$$\log y = x.$$

The logarithm of y will depend upon the value of y, and also upon a, the base of the system in which the logarithms are taken.

Let it be required to develop $\log y$ into a series arranged according to the ascending powers of y, with co-efficients that are independent of y and dependent upon a, the base of the system.

Let us first assume a development of the required form,

$$\log y = M + Ny + Py^2 + Qy^3 + \&c.,$$

in which M, N, P, &c. are independent of y, and dependent upon a. It is now required to find such values for these coefficients as will make the development true for every value of y.

Now, if we make $y = 0$, $\log y$ becomes infinite, and is either negative or positive, according as the base a is greater or less than 1, (Arts. 234 and 235). But the second member under this supposition, reduces to M, a finite number: hence, the development cannot be made under that form.

Again, assume,

$$\log y = My + Ny^2 + Py^3 + \&c.$$

If we make $y = 0$, we have

$$\log 0 = 0 \quad \text{that is,} \quad \pm \infty = 0,$$

which is absurd, and therefore the development cannot be made under the last form. Hence, we conclude that,

The logarithm of a number cannot be developed according to the ascending powers of that number.

Let us write $(1 + y)$, for y in the first member of the assumed development; we shall have,

$$\log (1 + y) = My + Ny^2 + Py^3 + Qy^4 + \&c. \quad - \quad - \quad (1),$$

making $y = 0$, the equation is reduced to $\log 1 = 0$, which does not present any absurdity.

Since equation (1) is true for any value of y, we may write z for y; whence,

$$\log (1 + z) = Mz + Nz^2 + Pz^3 + Qz^4 + \&c. \quad - \quad - \quad - \quad (2).$$

Subtracting equation (2) from equation (1), member from member, we obtain,

$$\log (1 + y) - \log (1 + z) = M(y - z) + N(y^2 - z^2) + P(y^3 - z^3)$$
$$+ Q(y^4 - z^4) \quad - \quad - \quad - \quad (3).$$

The second member of this equation is divisible by $(y-z)$, let us endeavor to place the first member under such a form that it shall also be divisible by $(y-z)$. We have,

$$\log(1+y) - \log(1+z) = \log\left(\frac{1+y}{1+z}\right) = \log\left(1+\frac{y-z}{1+z}\right).$$

But since $\frac{y-z}{1+z}$ can be regarded as a single quantity, we may substitute it for y in equation (1), which gives,

$$\log\left(1+\frac{y-z}{1+z}\right) = M\left(\frac{y-z}{1+z}\right) + N\left(\frac{y-z}{1+z}\right)^2 + P\left(\frac{y-z}{1+z}\right)^3 + \&c.$$

Substituting this development for its equal, in the first member of equation (3), and dividing both members of the resulting equation by $(y-z)$, and we have,

$$M\left(\frac{1}{1+z}\right) + N\frac{(y-z)}{(1+z)^2} + P\frac{(y-z)^2}{(1+z)^3} + \&c. = M + N(y+z)$$
$$+ P(y^2+yz+z^2) + \&c.$$

Since this equation is true for all values of y and z, make $z=y$, and there will result

$$\frac{M}{1+y} = M + 2Ny + 3Py^2 + 4Qy^3 + 5Ry^4 + \&c.$$

Clearing of fractions, and transposing, we obtain,

$$\left.\begin{array}{l}+M+2N\\-M+\ \ M\end{array}\right| y \left.\begin{array}{l}+3P\\+2N\end{array}\right| y^2 \left.\begin{array}{l}+4Q\\+3P\end{array}\right| y^3 \left.\begin{array}{l}+5R\\+4Q\end{array}\right| y^4 + \&c. = 0,$$

and since this equation is identical, we have,

$$M - \ M = 0; \quad \text{whence,} \quad M = M;$$

$$2N + \ M = 0; \quad \text{whence,} \quad N = -\frac{M}{2};$$

$$3P + 2N = 0; \quad \text{whence,} \quad P = -\frac{2N}{3} = \frac{M}{3};$$

$$4Q + 3P = 0; \quad \text{whence,} \quad Q = -\frac{3P}{4} = -\frac{M}{4}.$$

&c. &c.

The law of the co-efficients in the development is evident; the co-efficient of y^n is $\mp \frac{M}{n}$, according as n is even or odd.

Substituting these values for N, P, Q, &c., in equation (1), we find for the development of log $(1+y)$;

$$\log(1+y) = My - \frac{M}{2}y^2 + \frac{M}{3}y^3 - \frac{M}{4}y^4 \text{ . . \&c.}$$

$$= M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} \text{ . . \&c.}\right) \quad \text{- - (4,}$$

which is called the *logarithmic series.*

Hence, we see that the logarithm of a number may be developed into a series, according to the ascending powers of a number less than it by 1.

In the above development, the co-efficients have all been determined in terms of M. This should be so, since M depends upon the base of the system, and to the base any value may be assigned. By examining equation (4), we see that,

The expression for the logarithm of any number is composed of two factors, one dependent on the number, and the other on the base of the system in which the logarithm is taken.

The factor which depends on the base, is called the *modulus* of the system of logarithms.

237. If we take the logarithm of $1+y$ in a new system and denote it by $l(1+y)$, we shall have,

$$l(1+y) = M'\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \text{\&c.}\right) \quad \text{- - (5),}$$

in which M' is the modulus of the new system.

If we suppose y to have the same value in equations (4) and (5), and divide the former by the latter, member by member, we have $\frac{\log(1+y)}{l(1+y)} = \frac{M}{M'}$; whence, (Art. 183,)

$$l(1+y) : \log(1+y) :: M' : M; \quad \text{hence,}$$

The logarithms of the same number, taken in two different systems, are to each other as the moduli of those systems.

238. Having shown that the modulus and base of a system of logarithms are mutually dependent on each other, it follows, that if a value be assigned to one of them, the corresponding value of the other must be determined from it.

If then, we make the modulus

$$M' = 1,$$

the base of the system will assume a fixed value. The system of logarithms resulting from such a modulus, and such a base, is called the *Naperian System.* This was the first system known, and was invented by Baron Napier, a Scotch mathematician.

If we designate the Naperian logarithm by l, and the logarithm in any other system by log, the above proportion becomes,

$$l\,(1 + y) \;:\; \log\,(1 + y) \;::\; 1 \;:\; M;$$

whence,
$$M \times l\,(1 + y) = \log\,(1 + y).$$

Hence, we see that,

The Naperian logarithm of any number, multiplied by the modulus of any other system, will give the logarithm of the same number in that system.

The modulus of the Naperian System being 1, it is found most convenient to compare all other systems with the Naperian; and hence, the modulus of *any system* of logarithms, is

The number by which if the Naperian logarithm of any number be multiplied, the product will be the logarithm of the same number in that system.

239. Again, $\quad M \times l\,(1 + y) = \log\,(1 + y)$, gives

$$l\,(1 + y) = \frac{\log\,(1 + y)}{M}; \quad \text{that is,}$$

The logarithm of any number divided by the modulus of its system, is equal to the Naperian logarithm of the same number.

240. If we take the Naperian logarithm and make $y = 1$ equation (5) becomes,

$$l\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

a series which does not converge rapidly, and in which it would be necessary to take a great number of terms to obtain a near approximation. In general, this series will not serve for determining the logarithms of entire numbers, since for every number greater than 2 we should obtain a series in which the terms would go on increasing continually.

241. In order to deduce a logarithmic series sufficiently converging to be of use in computing the Naperian logarithms of numbers, let us take the logarithmic series and make $M' = 1$. Designating, as before, the Naperian logarithm by l, we shall have,

$$l(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \text{\&c.} \quad \cdots \quad (1).$$

If now, we write in equation (1), $-y$ for y, it becomes,

$$l(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} - \text{\&c.} \quad \cdots \quad (2)$$

Subtracting equation (2) from (1), member from member, we have,

$$l(1+y) - l(1-y) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \frac{y^9}{9} + \text{\&c.}\right) \cdots \quad (3).$$

But,

$$l(1+y) - l(1-y) = l\left(\frac{1+y}{1-y}\right); \quad \text{whence,}$$

$$l\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \frac{y^9}{9} + \text{\&c.}\right) \cdots \quad (4).$$

If now we make $\dfrac{1+y}{1-y} = \dfrac{z+1}{z}$, we shall have,

$$(1+y)z = (1-y)(z+1), \quad \text{whence,} \quad y = \frac{1}{2z+1}.$$

Substituting these values in equation (4), and observing that $l\left(\dfrac{z+1}{z}\right) = l(z+1) - lz$ we find,

$$l(z+1) - lz = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) (5),$$

or, by transposition,

$$l(z+1) = lz + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) \quad (6),$$

Let us make use of formula (6) to explain the method of computing a table of Naperian logarithms. It may be remarked, that it is only necessary to compute from the formula the logarithms of prime numbers; those of other numbers may be found by taking the sum of the logarithms of their factors.

The logarithm of 1 is 0. If now we make $z = 1$, we can find the logarithm of 2; and by means of this, if we make $z = 2$, we can find the logarithm of 3, and so on, as exhibited below.

$$l1 = 0 \quad \ldots\ldots\ldots\ldots \quad = 0.000000;$$

$$l2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} \ldots\right) \quad \ldots \quad = 0.693147;$$

$$l3 = 0.693147 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} \ldots\right) = 1.098612;$$

$$l4 = 2 \times l2 \quad \ldots\ldots\ldots\ldots \quad = 1.386294;$$

$$l5 = 1.386294 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} \ldots\right) = 1.609437;$$

$$l6 = l2 + l3 \quad \ldots\ldots\ldots\ldots \quad = 1.791759;$$

$$l7 = 1.791759 + 2\left(\frac{1}{13} + \frac{1}{3 \cdot (13)^3} + \frac{1}{5 \cdot (13)^5} + \ldots\right) = 1.945910;$$

$$l8 = l4 + l2 \quad \ldots\ldots\ldots\ldots \quad = 2.079441;$$

$$l9 = 2 \times l3 \quad \ldots\ldots\ldots\ldots \quad = 2.197224;$$

$$l10 = l5 + l2 \quad \ldots\ldots\ldots\ldots \quad = 2.302585;$$

&c. &c.

In like manner, we may compute the Naperian logarithms of all numbers. Other formulas may be deduced, which are

more rapidly converging than the one above given, but this serves to show the facility with which logarithms may be computed.

241*. We have already observed, that the *base* of the common system of logarithms is 10. We will now find its *modulus*. We have,

$$l(1+y) : \log(1+y) \; : \; : \; 1 : M \text{ (Art. 238).}$$

If we make $y = 9$, we shall have,

$$l\,10 : \quad \log 10 \quad : \; : \quad 1 : M.$$

But the $l\,10 = 2.302585093$, and $\log 10 = 1$ (Art. 228);

hence, $M = \dfrac{1}{2.302585093} = 0.434294482 =$ the modulus of the common system.

If now, we multiply the Naperian logarithms before found, by this modulus, we shall obtain a table of common logarithms (Art. 238).

All that now remains to be done, is to find the base of the Naperian system. If we designate that base by e, we shall have (Art. 237),

$$l\,e : \log e \; : \; : \; 1 : 0.434294482.$$

But $l\,e = 1$ (Art. 235): hence,

$$1 : \log e \; : \; : \; 1 : 0.434294482;$$

hence, $$\log e = 0.434294482.$$

But as we have already explained the method of calculating the common tables, we may use them to find the number whose logarithm is 0.434294482, which we shall find to be 2.718281828; hence,

$$e = 2.718281828\ldots\ldots$$

We see from the last equation but one, that

The modulus of the common system is equal to the common logarithm of the Naperian base.

Of Interpolation.

242. When the law of a series is given, and several terms taken at equal distances are known, we may, by means of the formula,

$$T = a + nd_1 + \frac{n(n-1)}{1 \cdot 2} d_2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_3 + \&c. \quad \text{- - -} \quad (1),$$

already deduced, (Art. 209), introduce other terms between them, which terms shall conform to the law of the series This operation is called *interpolation.*

In most cases, the law of the series is not given, but only numerical values of certain terms of the series, taken at fixed intervals; in this case we can only approximate to the law of the series, or to the value of any intermediate term, by the aid of formula (1).

To illustrate the use of formula (1) in interpolating a term in a tabulated series of numbers, let us suppose that we have the logarithms of 12, 13, 14, 15, and that it is required to find the logarithm of $12\frac{1}{2}$. Forming the orders of differences from the logarithms of 12, 13, 14 and 15 respectively, and taking the first terms of each,

12		13		14		15
1.079181,		1.113943,		1.146128,		1.176091,
	0.034762,		0.032185,		0.029963,	
		− 0.002577,		− 0.002222,		
			+ 0.000355,			

we find $\quad d_1 = 0.034762,\ d_2 = -0.002577,\ d_3 = 0.000355.$

If we consider log 12 as the first term, we have also

$$a = 1.079181 \quad \text{and} \quad n = \frac{1}{2}.$$

Making these several substitutions in the formula, and neglecting the terms after the fourth, since they are inappreciable we find,

$$T = a + \frac{1}{2} d_1 - \frac{1}{8} d_2 + \frac{1}{16} d_3 = \log 12\tfrac{1}{2}$$

or, by substituting for d_1, d_2, &c., their values, and for a its value,

a - - - - - - - -	1.079181
$\frac{1}{2} d_1$ - - - - - - - -	0.017381
$\frac{1}{8} d_2$ - - - - - - -	0.000322
$\frac{1}{16} d_3$ - - - - - - - -	0.000022
Log $12\frac{1}{2}$ - - - -	1.096906

Had it been required to find the logarithm of 12.39, we should have made $n = .39$, and the process would have been the same as above. In like manner we may interpolate terms between the tabulated terms of any mathematical table.

INTEREST.

243. The solution of all problems relating to interest, may be greatly simplified by employing algebraic formulas.

In treating of this subject, we shall employ the following notation:

Let p denote the amount bearing interest, called the *principal;*
r " the part of \$1, which expresses its interest for one year, called the *rate per cent.;*
t " the time, in years, that p draws interest;
i " the interest of p dollars for t years;
S " $p +$ the interest which accrues in the time t. This sum is called the *amount.*

Simple Interest.

To find the interest of a sum p *for* t *years, at the rate* r, *and the amount then due.*

Since r denotes the part of a dollar which expresses its interest for a single year, the interest of p dollars for the same

time will be expressed by pr; and for t years it will be t times as much: hence,

$$i = ptr \quad \text{- - - - - - -} \quad (1);$$

and for the amount due,

$$S = p + ptr = p(1 + tr) \quad \text{- -} \quad (2).$$

EXAMPLES.

1. What is the interest, and what the amount of \$365 for three years and a half, at the rate of 4 per cent. per annum. Here,

$$p = \$365;$$

$$r = \frac{4}{100} = 0.04;$$

$$t = 3.5;$$

$$i = ptr = 365 \times 3.5 \times 0.04 = \$51,10:$$

hence, $$S = 365 + 51,10 = \$416,10.$$

Present Value and Discount at Simple Interest.

The *present value* of any sum S, due t years hence, is the principal p, which put at interest for the time t, will produce the amount S.

The *discount* on any sum due t years hence, is the difference between that sum and the present value.

To find the present value of a sum of dollars denoted by S, *due* t *years hence, at simple interest, at the rate* r*;* also, *the discount.*

We have, from formula (2),

$$S = p + ptr;$$

and since p is the principal which in t years will produce the sum S, we have,

$$p = \frac{S}{1 + tr} \quad \text{- - - - - -} \quad (3);$$

and for the discount, which we will denote by D, we have

$$D = S - \frac{S}{1+tr} = \frac{Str}{1+tr} \quad \cdot \quad \cdot \quad \cdot \quad (4).$$

1. Required the discount on \$100, due 3 months hence, at the rate of $5\frac{1}{2}$ per cent. per annum.

$$S = \$100 \qquad = \$100,$$

$$t = 3 \text{ months} = 0.25.$$

$$r = \frac{5.5}{100} \qquad = .055.$$

Hence, the present value p is

$$p = \frac{S}{1+tr} = \frac{100}{1+.01375} = \$98{,}643\,;$$

hence, $\quad D = S - p = 100 - 98{,}643 = \$1{,}357.$

Compound Interest.

Compound interest is when the interest on a sum of money becoming due, and not paid, is added to the principal, and the interest then calculated on this amount as on a new principal.

To find the amount of a sum p *placed at interest for* t *years, compound interest being allowed annually at the rate* r.

At the end of one year the amount will be,

$$S = p + pr = p(1+r).$$

Since compound interest is allowed, this sum now becomes the principal, and hence, at the end of the second year, the amount will be,

$$S' = p(1+r) + pr(1+r) = p(1+r)^2.$$

Regard $p(1+r)^2$ as a new principal; we have, at the end of the third year,

$$S'' = p(1+r)^2 + pr(1+r)^2 = p(1+r)^3;$$

and at the end of t years,

$$S = p(1 + r)^t \quad . \quad . \quad . \quad . \quad (5).$$

And from Articles 230 and 232, we have,

$$\log S = \log p + t \log (1 + r);$$

and if any three of the four quantities S, p, t, and r, are given, the remaining one can be determined.

Let it be required to find the time in which a sum p will double itself at compound interest, the rate being 4 per cent. per annum.

We have, from equation (5),

$$S = p(1 + r)^t.$$

But by the conditions of the question,

$$S = 2p = p(1 + r)^t:$$

hence,
$$2 = (1 + r)^t.$$

and
$$t = \frac{\log 2}{\log (1 + r)} = \frac{0.301030}{0.017033},$$

$$= 17.673 \text{ years},$$

$$= 17 \text{ years, } 8 \text{ months, } 2 \text{ days}.$$

To find the Discount.

The discount being the difference between the sum S and p, we have,

$$D = S - \frac{S}{(1 + r)^t} = S\left(1 - \frac{1}{(1 + r)^t}\right)$$

CHAPTER X.

GENERAL THEORY OF EQUATIONS.

244. Every equation containing but one unknown quantity which is of the m^{th} degree, m being any positive whole number, may, by transposing all its terms to the first member and dividing by the co-efficient of x^m, be reduced to the form

$$x^m + Px^{m-1} + Qx^{m-2} + \quad . \quad . \quad . \quad . \quad + Tx + U = 0.$$

In this equation P, Q, ... T, U, are co-efficients in the most general sense of the term; that is, they may be positive or negative, entire or fractional, real or imaginary.

The last term U is the co-efficient of x^0, and is called the *absolute term.*

If none of these co-efficients are 0, the equation is said to be *complete;* if any of them are 0, the equation is said to be *incomplete.*

In discussing the properties of equations of the m^{th} degree, involving but one unknown quantity, we shall hereafter suppose them to have been reduced to the form just given.

245. We have already defined the *root* of an equation (Art. 77) to be *any expression, which, when substituted for the unknown quantity in the equation, will satisfy it.*

We have shown that every equation of the first degree has one root, that every equation of the second degree has two roots; and in general, if the two members of an equation are equal, they must be so for at least *some one value* of the

unknown quantity, either real or imaginary. Such value of the unknown quantity is a *root of the equation*: hence, we infer, that *every equation, of whatever degree, has at least one root.*

We shall now demonstrate some of the principal properties of equations of any degree whatever.

First Property.

246. In every equation of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots\ldots + Tx + U = 0,$$

if a *is a root, the first member is divisible by* x — a; and conversely, *if the first member is divisible by* x — a, a *is a root of the equation.*

Let us apply the rule for the division of the first member by $x - a$, and continue the operation till a remainder is found which is independent of x; that is, which does not contain x.

Denote this remainder by R and represent the quotient found by Q', and we shall have,

$$x^m + Px^{m-1} \ldots\ldots + Tx + U = Q'(x - a) + R.$$

Now, since by hypothesis, a is a root of the equation, if we substitute a for x, the first member of the equation will reduce to zero; the term $Q'(x - a)$ will also reduce to 0, and consequently, we shall have

$$R = 0.$$

But since R does not contain x, its value will not be affected by attributing to x the particular value a: hence, the remainder R is equal to 0, whatever may be the value of x, and consequently, the first member of the equation

$$x^m + Px^{m-1} + Qx^{m-2} \ldots\ldots + Tx + U = 0,$$

is exactly divisible by $x - a$.

Conversely, if $x - a$ is an exact divisor of the first member of the equation, the quotient Q' will be exact, and we shall have $R = 0$: hence,

$$x^m + Px^{m-1} \ldots\ldots + Tx + U = Q'(x - a).$$

If now, we suppose $x = a$, the second member will reduce to zero, consequently, the first will reduce to zero, and hence a will be a root of the equation (Art. 245). It is evident, from the nature of division, that the quotient Q' will be of the form

$$x^{m-1} + P'x^{m-2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad + R'x + U' = 0.$$

247. It follows from what has preceded, that in order to discover whether any polynomial is exactly divisible by the binomial $x - a$, it is sufficient to see if the substitution of a for x will reduce the polynomial to zero.

Conversely, if any polynomial is exactly divisible by $x - a$, then we know, that if the polynomial be placed equal to zero, a will be a root of the resulting equation.

The property which we have demonstrated above, enables us to diminish the degree of an equation by 1 when we know one of its roots, by a simple division; and if two or more roots are known, the degree of the equation may be still further diminished by successive divisions.

EXAMPLES.

1. A root of the equation,

$$x^4 - 25x^2 + 60x - 36 = 0,$$

is 3: what does the equation become when freed of this root?

$$\begin{array}{rrrr|l}
x^4 & -25x^2 & +60x & -36 & x - 3 \\
\cline{5-5}
x^4 - 3x^3 & & & & x^3 + 3x^2 - 16x + 12. \\
\cline{1-2}
+3x^3 & -25x^2 & & & \\
3x^3 & -9x^2 & & & \\
\cline{1-3}
& -16x^2 & +60x & & \\
& -16x^2 & +48x & & \\
\cline{2-4}
& & 12x & -36 & \\
& & 12x & -36 & \\
\cline{3-4}
\end{array}$$

Ans. $x^3 + 3x^2 - 16x + 12 = 0.$

2. Two roots of the equation,

$$x^4 - 12x^3 + 48x^2 - 68x + 15 = 0,$$

are 3 and 5: what does the equation become when freed of them? *Ans.* $x^2 - 4x + 1 = 0$

3. A root of the equation,

$$x^3 - 6x^2 + 11x - 6 = 0,$$

is 1: what is the reduced equation?

Ans. $x^2 - 5x + 6 = 0$.

4. Two roots of the equation,

$$4x^4 - 14x^3 - 5x^2 + 31x + 6 = 0,$$

are 2 and 3: find the reduced equation.

Ans. $4x^2 + 6x + 1 = 0$.

Second Property.

248. *Every equation involving but one unknown quantity, has as many roots as there are units in the exponent which denotes its degree, and no more.*

Let the proposed equation be

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0.$$

Since every equation is known to have at least one root (Art. 245), if we denote that root by a, the first member will be divisible by $x - a$, and we shall have the equation,

$$x^m + Px^{m-1} + \ldots = (x - a)(x^{m-1} + P'x^{m-2} + \ldots) \text{ - - - - } (1).$$

But if we place,

$$x^{m-1} + P'x^{m-2} + \ldots = 0,$$

we obtain a new equation, which has at least one root.

Denote this root by b, and we have (Art. 246),

$$x^{m-1} + P'x^{m-2} + \ldots = (x - b)(x^{m-2} + P''x^{m-3} + \ldots).$$

Substituting the second member, for its value, in equation (1), we have,

$$x^m + Px^{m-1} + \ldots = (x - a)(x - b)(x^{m-2} + P''x^{m-3} + \ldots) \cdot\cdot (2).$$

Reasoning upon the polynomial,

$$x^{m-2} + P''x^{m-3} + \ldots,$$

as upon the preceding polynomial, we have

$$x^{m-2} + P''x^{m-3} + \ldots = (x - c)(x^{m-3} + P'''x^{m-4} + \ldots),$$

and by substitution,

$$x^m + Px^{m-1} + \ldots = (x - a)(x - b)(x - c)(x^{m-3} + P'''x^{m-4}) \text{ - - - } (3).$$

By continuing this operation, we see that for each binomial factor of the first degree with reference to x, that we separate, the degree of the polynomial factor is reduced by 1; therefore, after $m-2$ binomial factors have been separated, the polynomial factor will become of the second degree with reference to x, which can be decomposed into two factors of the first degree (Art. 115), of the form $x-k$, $x-l$.

Now, supposing the $m-2$ factors of the first degree to have already been indicated, we shall have the identical equation,

$$x^m + Px^{m-1} + \ldots = (x-a)(x-b)(x-c)\ldots(x-k)(x-l) = 0;$$

from which we see, that the *first member of the proposed equation may be decomposed into* m *binomial factors of the first degree.*

As there is *a root* corresponding to each binomial factor of the first degree (Art. 246), it follows that the m binomial factors of the first degree, $x-a$, $x-b$, $x-c \ldots\ldots$, give the m roots, a, b, $c \ldots$, of the proposed equation.

But the equation can have no other roots than $a, b, c \ldots k, l$. For, if it had a root a', different from $a, b, c \ldots\ldots l$, it would have a divisor $x-a'$, different from $x-a$, $x-b$, $x-c \ldots x-l$, which is impossible; therefore,

Every equation of the m^{th} degree has m roots, and can have no more.

249. In equations which arise from the multiplication of equal factors, such as

$$(x-a)^4(x-b)^3(x-c)^2(x-d) = 0,$$

the number of roots is *apparently* less than the number of units in the exponent which denotes the degree of the equation. But this is not really so; for the above equation actually has ten roots, four of which are equal to a, three to b, two to c, and one to d.

It is evident that no quantity a', different from a, b, c, d, can verify the equation; for, if it had a root a', the first member would be divisible by $x-a'$, which is impossible.

Consequence of the Second Property.

250. It has been shown that the first member of every equation of the m^{th} degree, has m binomial divisors of the first degree, of the form

$$x-a, \quad x-b, \quad x-c, \ldots x-k, \quad x-l.$$

If we multiply these divisors together, *two and two*, *three and three*, &c., we shall obtain as many divisors of the second, third, &c. degree, with reference to x, as we can form different combinations of m quantities, taken two and two, three and three, &c. Now, the number of these combinations is expressed by

$$m \cdot \frac{m-1}{2}, \quad m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \ldots \text{ (Art. 132)};$$

hence, the proposed equation has

$$m \cdot \frac{m-1}{2}$$

divisors of the second degree;

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}$$

divisors of the third degree;

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}$$

divisors of the fourth degree; and so on.

Composition of Equations.

251. If we resume the identical equation of Art. 248,

$$x^m + Px^{m-1} + Qx^{m-2} \ldots + U = (x-a)(x-b)(x-c) \ldots (x-l) \ldots$$

and suppose the multiplications indicated in the second member to be performed, we shall have, from the law demonstrated in article 135, the following relations:

$$P = -a-b-c- \ldots -k-l, \text{ or } -P = a+b+c+ \ldots +k+l,$$

$$Q = ab+ac+bc+ \ldots ak+kl,$$

$$R = -abc-abd-bcd \ldots -ikl, \text{ or } -R = abc+abd+ \ldots + ikl,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$U = \pm abcd \ldots ikl, \quad \text{or} \quad \pm U = abc \ldots ikl.$$

The double sign has been placed before the product of a, b, c, &c. in the last equation, since the product $-a \times -b \times -c \,.\,. \times -l$, will be *plus* when the degree of the equation is *even*, and *minus* when it is odd.

By considering these relations, we derive the following conclusions with reference to the values of the co-efficients:

1st. *The co-efficient of the second term, with its sign changed, is equal to the algebraic sum of the roots of the equation.*

2d. *The co-efficient of the third term is equal to the sum of the different products of the roots, taken two in a set.*

3d. *The co-efficient of the fourth term, with its sign changed, is equal to the sum of the different products of the roots, taken three in a set, and so on.*

4th. *The absolute term, with its sign changed when the equation is of an odd degree, is equal to the continued product of all the roots of the equation.*

Consequences.

1. If one of the roots of an equation is 0, there will be no absolute term; and conversely, if there is no absolute term, one of the roots must be 0.

2. If the co-efficient of the second term is 0, the numerical sum of the positive roots is equal to that of the negative roots.

3. Every root will exactly divide the absolute term.

It will be observed that the properties of equations of the second degree, already demonstrated, conform in all respects to the principles demonstrated in this article.

EXAMPLES OF THE COMPOSITION OF EQUATIONS.

1. Find the equation whose roots are 2, 3, 5, and -6.

We have, from the principles already established, the equation

$$(x-2)(x-3)(x-5)(x+6)=0;$$

whence, by the application of the preceding principles, we obtain the equation,

$$x^4-4x^3-29x^2+156x-180=0.$$

2. What is the equation whose roots are 1, 2, and -3?

Ans. $x^3 - 7x + 6 = 0$.

3. What is the equation whose roots are 3, -4, $2 + \sqrt{3}$, nd $2 - \sqrt{3}$? *Ans.* $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$.

4. What is the equation whose roots are $3 + \sqrt{5}$, $3 - \sqrt{5}$, and -6? *Ans.* $x^3 - 32x + 24 = 0$.

5. What is the equation whose roots are 1, -2, 3, -4, 5, and -6?

Ans. $x^6 + 3x^5 - 41x^4 - 87x^3 + 400x^2 + 444x - 720 = 0$.

6. What is the equation whose roots are . . . . $2 + \sqrt{-1}$, $2 - \sqrt{-1}$, and -3? *Ans.* $x^3 - x^2 - 7x + 15 = 0$

Greatest Common Divisor.

252. The principle of the greatest common divisor is of frequent application in discussing the nature and properties of equations, and before proceeding further, it is necessary to investigate a rule for determining the greatest common divisor of two or more polynomials.

The greatest common divisor of two or more polynomials is the greatest algebraic expression, with respect both to co-efficients and exponents, that will exactly divide them.

A polynomial is *prime*, when no other expression except 1 will exactly divide it.

Two polynomials are *prime with respect to each other*, when they have no common factor except 1.

253. Let A and B designate any two polynomials arranged with reference to the same leading letter, and suppose the polynomial A to contain the highest exponent of the leading letter. Denote the greatest common divisor of A and B by D, and let the quotients found by dividing each polynomial by D

be represented by A' and B' respectively. We shall then have the equations,

$$\frac{A}{D} = A', \quad \text{and} \quad \frac{B}{D} = B';$$

whence, $$A = A' \times D \quad \text{and} \quad B = B' \times D.$$

Now, D contains all the factors common to A and B. For, if it does not, let us suppose that A and B have a common factor d which does not enter D, and let us designate the quotients of A' and B', by this factor, by A'' and B''. We shall then have,

$$A = A''.d.D \quad \text{and} \quad B = B''.d.D;$$

or, by division,

$$\frac{A}{d.D} = A'' \quad \text{and} \quad \frac{B}{d.D} = B''.$$

Since A'' and B'' are entire, both A and B are divisible by $d.D$, which must be greater than D, either with respect to its co-efficients or its exponents; but this is absurd, since, by hypothesis, D is the greatest common divisor of A and B. Therefore, D contains all the factors common to A and B.

Nor can D contain any factor which is not common to A and B. For, suppose D to have a factor d' which is not con tained in A and B, and designate the other factor of D by D'; we shall have the equations,

$$A = A'.d'.D' \quad \text{and} \quad B = B'.d'.D';$$

or, dividing both members of these equations by d',

$$\frac{A}{d'} = A'.D' \quad \text{and} \quad \frac{B}{d'} = B'.D'.$$

Now, the second members of these two equations being entire, the first members must also be entire; that is, both A and B are divisible by d' and therefore the supposition that d' is not a common factor of A and B is absurd. Hence,

1st. *The greatest common divisor of two polynomials contains all the factors common to the polynomials, and does not contain any other factors.*

254. If, now, we apply the rule for dividing A by B, and continue the process till the greatest exponent of the leading letter in the remainder is at least one less than it is in the polynomial B, and if we designate the remainder by R, and the quotient found, by Q, we shall have,

$$A = B \times Q + R \quad - \; - \; - \; - \quad (1).$$

If, as before, we designate the greatest common divisor of A and B by D, and divide both members of the last equation by it, we shall have,

$$\frac{A}{D} = \frac{B}{D} \times Q + \frac{R}{D}.$$

Now, the first member of this equation is an entire quantity, and so is the first term of the second member; hence $\frac{R}{D}$ must be entire; which proves that the greatest common divisor of A and B also divides R.

If we designate the greatest common divisor of B and R by D', and divide both members of equation (1) by it, we shall have,

$$\frac{A}{D'} = \frac{B}{D'} \times Q + \frac{R}{D'}.$$

Now, since by hypothesis D' is a common divisor of B and R, both terms of the second member of this equation are entire; hence, the first member must be entire; which proves that the greatest common divisor of B and R, also divides A.

We see that D', the greatest common divisor of B and R, cannot be less than D, since D divides both B and R; nor can D, the greatest common divisor of A and B, be less than D', because D' divides both A and B; and since neither can be less than the other, they must be equal; that is, $D = D'$. Hence,

2d. *The greatest common divisor of two polynomials, is the same as that between the second polynomial and their remainder after division.*

From the principle demonstrated in Art. 253, we see that we may multiply or divide one polynomial by any factor that is

not contained in the other, without affecting their greatest common divisor.

255. From the principles of the two preceding articles, we deduce, for finding the greatest common divisor of two polynomials, the following

RULE.

I. *Suppress the monomial factors common to all the terms of the first polynomial; do the same with the second polynomial; and if the factors so suppressed have a common divisor, set it aside, as forming a factor of the common divisor sought.*

II. *Prepare the first polynomial in such a manner that its first term shall be divisible by the first term of the second polynomial, both being arranged with reference to the same letter: Apply the rule for division, and continue the process till the greatest exponent of the leading letter in the remainder is at least one less than it is in the second polynomial. Suppress, in this remainder, all the factors that are common to the co-efficients of the different powers of the leading letter; then take this result as a divisor and the second polynomial as a dividend, and proceed as before.*

III. *Continue the operation until a remainder is obtained which will exactly divide the preceding divisor; this last remainder, multiplied by the factor set aside, will be the greatest common divisor sought; if no remainder is found which will exactly divide the preceding divisor, then the factor set aside is the greatest common divisor sought.*

EXAMPLES.

1. Find the greatest common divisor of the polynomials

$$a^3 - a^2b + 3ab^2 - 3b^3, \quad \text{and} \quad a^2 - 5ab + 4b^2.$$

First Operation. *Second Operation.*

$$\begin{array}{l|l}
a^3 - a^2b + 3ab^2 - 3b^3 & a^2 \quad 5ab + 4b^2 \\
\hline
\quad 4a^2b - \quad ab^2 - 3b^3 & a + 4b \\
\hline
\text{1st rem.} \quad 19ab^2 - 19b^3 & \\
\text{or,} \qquad 19b^2(a - b). &
\end{array}
\qquad
\begin{array}{r|l}
a^2 - 5ab + 4b^2 & a - b \\
\hline
-4ab + 4b^2 & a - 4b \\
\hline
0. &
\end{array}$$

Hence, $a - b$ is the greatest common divisor.

We begin by dividing the polynomial of the highest degree by that of the lowest; the quotient is, as we see in the above table, $a + 4b$, and the remainder $19ab^2 - 19b^3$.

But, $$19ab^2 - 19b^3 = 19b^2(a - b).$$

Now, the factor $19b^2$, will divide this remainder without dividing

$$a^2 - 5ab + 4b^2:$$

hence, the factor must be suppressed, and the question is reduced to finding the greatest common divisor between

$$a^2 - 5ab + 4b^2 \quad \text{and} \quad a - b.$$

Dividing the first of these two polynomials by the second, there is an exact quotient, $a - 4b$, hence, $a - b$ is the greatest common divisor of the two given polynomials. To verify this, let each be divided by $a - b$.

2. Find the greatest common divisor of the polynomials,

$$3a^5 - 5a^3b^2 + 2ab^4 \quad \text{and} \quad 2a^4 - 3a^2b^2 + b^4.$$

We first suppress a, which is a factor of each term of the first polynomial: we then have,

$$3a^4 - 5a^2b^2 + 2b^4 \,||\, 2a^4 - 3a^2b^2 + b^4.$$

We now find that the first term of the dividend will not contain the first term of the divisor. We therefore multiply the dividend by 2, which merely introduces into the dividend a factor not common to the divisor, and hence does not affect the common divisor sought. We then have,

$$\begin{array}{r|l}
6a^4 - 10a^2b^2 + 4b^4 & 2a^4 - 3a^2b^2 + b^4 \\ \cline{2-2}
6a^4 - 9a^2b^2 + 3b^4 & 3 \\ \hline
- a^2b^2 + b^4 & \\
- b^2(a^2 - b^2). &
\end{array}$$

We find after division, the remainder $-a^2b^2 + b^4$ which we put under the form $-b^2(a^2 - b^2)$. We then suppress $-b^2$, and divide,

$$\begin{array}{r|l}
2a^4 - 3a^2b^2 + b^4 & a^2 - b^2 \\ \cline{2-2}
2a^4 - 2a^2b^2 & 2a^2 - b^2 \\ \hline
- a^2b^2 + b^4 & \\
- a^2b^2 + b^4. & \\ \hline
\end{array}$$

Hence, $a^2 - b^2$ is the greatest common divisor.

3. Let it be required to find the greatest common divisor between the two polynomials,

$$-3b^3 + 3ab^2 - a^2b + a^3, \quad \text{and} \quad 4b^2 - 5ab + a^2.$$

First Operation.

$$\begin{array}{ll|l}
 & -12b^3 + 12ab^2 - 4a^2b + 4a^3 & 4b^2 - 5ab + a^2 \\ \hline
\text{1st rem.} & \quad - \; - \; - 3ab^2 - a^2b + 4a^3 & -3b, \quad -3a \\
 & \qquad -12ab^2 - 4a^2b + 16a^3 & \\ \hline
\text{2d rem.} & \qquad\qquad -19a^2b + 19a^3 & \\
\text{or,} & \qquad\qquad 19a^2(-b + a). &
\end{array}$$

Second Operation.

$$\begin{array}{r|l}
4b^2 - 5ab + a^2 & -b + a \\ \hline
-ab + a^2 & -4b + a \\ \hline
0. &
\end{array}$$

Hence, $-b + a$, or $a - b$, is the greatest common divisor

In the first operation we meet with a difficulty in dividing the two polynomials, because the first term of the dividend is not exactly divisible by the first term of the divisor. But if we observe that the co-efficient 4, is not a factor of all the terms of the polynomial

$$4b^2 - 5ab + a^2,$$

and therefore, by the first principle, that 4 cannot form a part of the greatest common divisor, we can, without affecting this common divisor, introduce this factor into the dividend. This gives,

$$-12b^3 + 12ab^2 - 4a^2b + 4a^3,$$

and then the division of the terms is possible.

Effecting this division, the quotient is $-3b$, and the remainder is,

$$-3ab^2 - a^2b + 4a^3.$$

As the exponent of b in this remainder is still equal to that of b in the divisor, the division may be continued, by multiplying this remainder by 4, in order to render the division of the first term possible. This done, the remainder becomes

$$-12ab^2 - 4a^2b + 16a^3;$$

which, divided by $4b^2 - 5ab + a^2$, gives the quotient $-3a$, which should be separated from the first by a comma, having no connexion with it. The remainder after this division, is

$$-19a^2b + 19a^3.$$

Placing this last remainder under the form $19a^2(-b + a)$, and suppressing the factor $19a^2$, as forming no part of the common divisor, the question is reduced to finding the greatest common divisor between

$$4b^2 - 5ab + a^2 \quad \text{and} \quad -b + a.$$

Dividing the first of these polynomials by the second, we obtain an exact quotient, $-4b + a$: hence, $-b + a$, or $a - b$, is the greatest common divisor sought.

256. In the above example, as in all those in which the exponent of the leading letter is greater by 1 in the dividend than in the divisor, we can abridge the operation by first multiplying every term of the dividend by the square of the co-efficient of the first term of the divisor. We can easily see that by this means, the first term of the quotient obtained will contain the first power of this co-efficient. Multiplying the divisor by the quotient, and making the reductions with the dividend thus prepared, the result will still contain the co-efficient as a factor, and the division can be continued until a remainder is obtained of a lower degree than the divisor, with reference to the leading letter.

Take the same example as before, viz.:

$$-3b^3 + 3ab^2 - a^2b + a^3 \quad \text{and} \quad 4b^2 - 5ab + a^2,$$

and multiply the dividend by $4^2 = 16$; and we have

First Operation.

$$\begin{array}{r|l} -48b^3 + 48ab^2 - 16a^2b + 16a^3 & 4b^2 - 5ab + a^2 \\ \hline -12ab^2 - 4a^2b + 16a^3 & -12b - 3a \\ \hline \end{array}$$

1st remainder, $\qquad -19a^2b + 19a^3$

or, $\qquad 19a^2(-b + a)$.

Second Operation.

$$\begin{array}{r|l} 4b^2 - 5ab + a^2 & -\ b + a \\ \hline -\ ab + a^2 & -4b + a \\ \hline \end{array}$$

2d remainder, $\qquad -0.$

When the exponent of the leading letter in the dividend exceeds that of the same letter in the divisor by two, three, &c., multiply the dividend by the third, fourth, &c. power of the co-efficient of the first term of the divisor. It is easy to see the reason of this.

257. It may be asked if the suppression of the factors, common to all the terms of one of the remainders, is *absolutely necessary*, or whether the object is merely to render the operations more simple. It will easily be perceived that the suppression of these factors *is necessary;* for, if the factor $19a^2$ was not suppressed in the preceding example, it would be necessary to multiply the whole dividend by this factor, in order to render its first term divisible by the first term of the divisor; but, then, a factor would be introduced into the dividend which is also contained in the divisor; and, consequently, the required greatest common divisor would contain the factor $19a^2$ which should form no part of it.

258. For another example, let it be required to find the greatest common divisor of the two polynomials,

$$a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4 \quad \text{and} \quad 4a^2b + 2ab^2 - 2b^3,$$

or simply of,

$$a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4 \quad \text{and} \quad 2a^2 + ab - b^2,$$

since the factor $2b$ can be suppressed, being a factor of the second polynomial and not of the first.

First Operation.

$$\begin{array}{r|l} 8a^4 + 24a^3b + 32a^2b^2 - 48ab^3 + 16b^4 & 2a^2 + ab - b^2 \\ \hline +20a^3b + 36a^2b^2 - 48ab^3 + 16b^4 & 4a^2 + 10ab + 13b^2 \\ \hline +26a^2b^2 - 38ab^3 + 16b^4 & \\ \hline \end{array}$$

1st remainder, $\qquad -51ab^3 + 29b^4$

or, $\qquad -b^3(51a - 29b).$

Second Operation.

Multiply by 2601, the square of 51.

$$
\begin{array}{lr|l}
 & 5202a^2 + 2601ab - 2601b^2 & 51a - 29b \\
 & 5202a^2 - 2958ab & 102a + 109b \\
\text{1st remainder,} & + 5559ab - 2601b^2 & \\
 & 5559ab - 3161b^2 & \\
\text{2d remainder,} & + 560b^2. &
\end{array}
$$

The exponent of the letter a in the dividend, exceeding that of the same letter in the divisor, by *two*, the whole dividend is multiplied by $2^3 = 8$. This done, we perform the division, and obtain for the first remainder,

$$- 51ab^3 + 29b^4.$$

Suppressing $-b^3$, this remainder becomes $51a - 29b$; and the new dividend is

$$2a^2 + ab - b^2.$$

Multiplying the dividend by $(51)^2 = 2601$, then effecting the division, we obtain for the second remainder $+ 560b^2$. Now, it results from the second principle (Art. 254), that the greatest common divisor must be a factor of the remainder after each division; therefore it should divide the remainder $560b^2$. But this remainder is *independent* of the leading letter a: hence, if the two polynomials have a common divisor, it must be *independent* of a, and will consequently be found as a factor in the co-efficients of the different powers of this letter, in each of the proposed polynomials. But it is evident that the co-efficients of these powers have not a common factor. Hence, *the two given polynomials are prime with respect to each other.*

259. The rule for finding the greatest common divisor of two polynomials, may readily be extended to three or more polynomials. For, having the polynomials A, B, C, D, &c., if we find the greatest common divisor of A and B, and then the greatest common divisor of this result and C, the divisor so ob

tained will evidently be the greatest common divisor of A, B, and C; and the same process may be applied to the remaining polynomials.

260. It often happens, after suppressing the monomial factors common to all the terms of the given polynomials, and arranging the remaining polynomials with reference to a particular letter, that there are polynomial factors common to the co-efficients of the different powers of the leading letter in one or both polynomials. In that case we suppress those factors in both, and if the suppressed factors have a common divisor, we set it aside, as forming a factor of the common divisor sought.

EXAMPLE.

Let it be required to find the greatest common divisor of the two polynomials

$$a^2d^2 - c^2d^2 - a^2c^2 + c^4, \quad \text{and} \quad 4a^2d - 2ac^2 + 2c^3 - 4acd.$$

The second contains a monomial factor 2. Suppressing it, and arranging the polynomials with reference to d, we have

$$(a^2 - c^2)\,d^2 - a^2c^2 + c^4, \quad \text{and} \quad (2a^2 - 2ac)\,d - ac^2 + c^3.$$

By considering the co-efficients, $a^2 - c^2$ and $-a^2c^2 + c^4$, in the first polynomial, it will be seen that $-a^2c^2 + c^4$ can be put under the form $-c^2(a^2 - c^2)$: hence, $a^2 - c^2$ is a common factor of the co-efficients in the first polynomial. In like manner, the co-efficients in the second, $2a^2 - 2ac$ and $-ac^2 + c^3$, can be reduced to $2a(a - c)$ and $-c^2(a - c)$; therefore, $a - c$ is a common factor of these co-efficients.

Comparing the two factors $a^2 - c^2$ and $a - c$, we see that the last will divide the first; hence, it follows that $a - c$ is a common factor of the proposed polynomials, and it is therefore a factor of the greatest common divisor.

Suppressing $a^2 - c^2$ in the first polynomial, and $a - c$ in the second, we obtain the two polynomials,

$$d^2 - c^2 \quad \text{and} \quad 2ad - c^2.$$

to which the ordinary process may be applied.

$$\begin{array}{l|l} \quad d^2 - c^2 & 2ad - c^2 \\ \cline{2-2} \quad 4a^2d^2 - 4a^2c^2 & 2ad + c^2 \\ \hline + 2ac^2d - 4a^2c^2 & \\ \hline - 4a^2c^2 + c^4. & \end{array}$$

After having multiplied the dividend by $4a^2$, and performed the division, we obtain a remainder $-4a^2c^2 + c^4$, independent of the letter d: hence, the two polynomials, $d^2 - c^2$ and $2ad - c^2$, are prime with respect to each other. Therefore, the greatest common divisor of the proposed polynomials is $a - c$.

261. It sometimes happens that one of the polynomials contains a letter which is not contained in the other.

In this case, it is evident that the greatest common divisor is independent of this letter. Hence, by arranging the polynomial which contains it, with reference to this letter, *the required common divisor will be the same as that which exists between the co-efficients of the different powers of the principal letter and the second polynomial.*

By this method we are led, it is true, to determine the greatest common divisor between three or more polynomials. But they will be more simple than the proposed polynomials. It often happens, that some of the co-efficients of the arranged polynomial are monomials, or, that we can discover by simple inspection that they are prime with respect to each other; and, in this case, we are certain that the proposed polynomials are prime with respect to each other.

Thus, in the example of the last article, after having suppressed the common factor $a - c$, which gives the results,

$$d^2 - c^2 \quad \text{and} \quad 2ad - c^2,$$

we know immediately that these two polynomials are prime with respect to each other; for, since the letter a is contained in the second and not in the first, it follows from what has just been said, that the common divisor must be contained in the co-efficients $2d$

and $-c^2$; but these are prime with respect to each other, and consequently, the expressions $d^2 - c^2$ and $2ad - c^2$, are also prime with respect to each other.

Let it be required to find the greatest common divisor of the two polynomials,

$$3bcq + 30mp + 18bc + 5mpq,$$

and,
$$4adq - 42fg + 24ad - 7fgq.$$

Now, the letter b is found in the first polynomial and not in the second. If then, we arrange the first with reference to b, we have,

$$(3cq + 18c)\,b + 30mp + 5mpq,$$

and the required greatest common divisor will be the same as that which exists between the second polynomial and the two co-efficients of b, which are,

$$3cq + 18c \quad \text{and} \quad 30mp + 5mpq.$$

Now, the first of these co-efficients can be put under the form $3c(q + 6)$, and the other becomes $5mp(q + 6)$; hence, $q + 6$ is a common factor of these co-efficients. It will therefore be sufficient to ascertain whether $q + 6$ is a factor of the second polynomial.

Arranging this polynomial with reference to q, it becomes

$$(4ad - 7fg)q - 42fg + 24ad;$$

and as the second part, $24ad - 42fg = 6(4ad - 7fg)$, it follows that this polynomial is divisible by $q + 6$, and gives the quotient $4ad - 7fg$. Therefore, $q + 6$ is the greatest common divisor of the proposed polynomials.

EXAMPLES.

1. Find the greatest common divisor of the two polynomials

$$6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,$$

and
$$4x^4 + 2x^3 - 18x^2 + 3x - 5.$$

Ans $2x^3 - 4x^2 + x - 1$

2. Find the greatest common divisor of the polynomials

$$20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4,$$

and
$$15x^4 - 9x^3 + 47x^2 - 21x + 28.$$

Ans. $5x^2 - 3x + 4$.

3. Find the greatest common divisor of the two polynomials

$$5a^4b^2 + 2a^3b^3 + ca^2 - 3a^2b^4 + bca,$$

and
$$a^5 + 5a^3d - a^3b^2 + 5a^2bd.$$

Ans $a^2 + ab$.

Transformation of Equations.

262. The object of a transformation, is to change an equation from a given form to another, from which we can more readily determine the value of the unknown quantity.

First.

To change a given equation involving fractional co-efficients to another of the same general form, but having the co-efficients of all its terms entire

If we have an equation of the form

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots Tx + U = 0,$$

and make
$$x = \frac{y}{k};$$

in which y is a new unknown quantity, and k entirely arbitrary; we shall have, after substituting this value for x, and multiplying every term by k^m,

$$y^m + Pky^{m-1} + Qk^2y^{m-2} + Rk^3y^{m-3} + \ldots + Tk^{m-1}y + Uk^m = 0,$$

an equation in which the co-efficients of the different powers of y are equal to those of the same powers of x in the given equation, multiplied respectively by k^0, k^1, k^2, k^3, k^4, &c.

It is now required to assign such a value to k as will make the co-efficients of the different powers of y entire.

To illustrate, let us take, as a general example, the equation

$$x^4 + \frac{a}{b}x^3 + \frac{c}{d}x^2 + \frac{e}{f}x + \frac{g}{h} = 0,$$

which becomes, after substituting $\frac{y}{k}$ for x, and multiplying by k^4,

$$y^4 + \frac{ak}{b}y^3 + \frac{ck^2}{d}y^2 + \frac{ek^3}{f}y + \frac{gk^4}{h} = 0.$$

Now, there may be two cases—

1st. Where the denominators b, d, f, h, are prime with respect to each other. In this case, as k is altogether arbitrary, take $k = bdfh$, *the product of the denominators*, the equation will then become,

$$y^4 + adfh \,.\, y^3 + cb^2df^2h^2 \,.\, y^2 + eb^3d^3f^2h^3 \,.\, y + gb^4d^4f^4h^3 = 0,$$

in which the co-efficients of y are entire, and that of the first term is 1.

2d. When the denominators contain common factors, we shall evidently render the co-efficients entire, by making k equal to the least common multiple of all the denominators. But we can simplify still more, by giving to k such a value that $k^1, k^2, k^3, \ldots$ shall contain the prime factors which compose b, d, f, h, raised to powers at least equal to those which are found in the denominators.

Thus, the equation

$$x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{9000} = 0,$$

becomes

$$y^4 - \frac{5k}{6}y^3 + \frac{5k^2}{12}y^2 - \frac{7k^3}{150}y - \frac{13k^4}{9000} = 0,$$

after making $x = \frac{y}{k}$, and reducing the terms.

First, if we make $k = 9000$, which is a multiple of all the other denominators, it is clear that the co-efficients become entire numbers.

But if we decompose 6, 12, 150, and 9000, into their prime factors, we find,

$6 = 2 \times 3, \quad 12 = 2^2 \times 3, \quad 150 = 2 \times 3 \times 5^2, \quad 9000 = 2^3 \times 3^2 \times 5^3.$

and by making

$$k = 2 \times 3 \times 5,$$

the product of the different prime factors, we obtain

$$k^2 = 2^2 \times 3^2 \times 5^2, \quad k^3 = 2^3 \times 3^3 \times 5^3, \quad k^4 = 2^4 \times 3^4 \times 5^4;$$

whence we see that the values of k, k^2, k^3, k^4, contain the prime factors of 2, 3, 5, raised to powers at least equal to those which enter into 6, 12, 150, and 9000. Hence, making

$$k = 2 \times 3 \times 5,$$

is sufficient to make the denominators disappear. Substituting this value, the equation becomes

$$y^4 - \frac{5.2.3.5}{2.3}y^3 + \frac{5.2^2.3^2.5^2}{2^2.3}y^2 - \frac{7.2^3.3^3.5^3}{2.3.5^2}y - \frac{13.2^4.3^4.5^4}{2^3.3^2.5^3} = 0,$$

which reduces to

$$y^4 - 5.5y^3 + 5.3.5^2y^2 - 7.2^2.3^2.5y - 13.2.3^2.5 = 0;$$

or, $\quad y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0.$

Hence, we perceive the necessity of taking k as small a number as possible: otherwise, we should obtain a transformed equation, having its co-efficients very great, as may be seen by reducing the transformed equation resulting from the supposition $k = 9000$.

Having solved the transformed equation, and found the values of y, the corresponding values of x may be found from the equation,

$$x = \frac{y}{k},$$

by substituting for y and k their proper values.

EXAMPLES.

1. $$x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0.$$

Making $x = \frac{y}{6}$, and we have,

$$y^3 - 14y^2 + 11y - 75 = 0.$$

2. $$x^5 - \frac{13}{12}x^4 + \frac{21}{40}x^3 - \frac{32}{225}x^2 - \frac{43}{600}x - \frac{1}{800} = 0.$$

Making $\quad x = \frac{y}{2^2.3.5} = \frac{y}{60}$, and we have,

$$y^5 - 65y^4 + 1890y^3 - 30720y^2 - 928800y - 972000 = 0.$$

Second.

To make the second or any other term disappear from an equation.

263. The difficulty of solving an equation generally diminishes with the number of terms involving the unknown quantity.

Thus the equation

$$x^2 = q, \quad \text{gives immediately,} \quad x = \pm\sqrt{q},$$

while the complete equation

$$x^2 + 2px + q = 0,$$

requires preparation before it can be solved.

Now, any given equation can always be *transformed* into an incomplete equation, in which the second term shall be wanting.

For, let there be the general equation,

$$x^m + Px^{m-1} + Qx^{m-2} + \;.\;.\;.\; + Tx + U = 0.$$

Suppose $\qquad x = u + x',$

u being a new unknown quantity, and x' entirely *arbitrary*.

By substituting $u + x'$ for x, we obtain

$$(u + x')^m + P(u + x')^{m-1} + Q(u + x')^{m-2} \ldots + T(u + x') + U = 0.$$

Developing by the binomial formula, and arranging with reference to u, we have

$$\left.\begin{array}{l|l|l}
u^m + mx' & u^{m-1} + m\,.\,\dfrac{m-1}{2}x'^2 & u^{m-2} + \ldots + x'^m \\
+P & \quad +(m-1)Px' & \qquad +Px'^{m-1} \\
 & \quad +Q & \qquad +Qx'^{m-2} \\
 & & \qquad + \;.\;.\;. \\
 & & \qquad .\;.\;.\;.\;. \\
 & & \qquad +Tx' \\
 & & \qquad +U
\end{array}\right\} = 0.$$

Since x' is entirely arbitrary, we may dispose of it in such way that we shall have

$$mx' + P = 0; \quad \text{whence,} \quad x' = -\frac{P}{m}.$$

Substituting this value of x' in the last equation, we shall obtain an incomplete equation of the form,

$$u^m + Q'u^{m-2} + R'u^{m-3} + \ldots T'u + U' = 0,$$

in which the second term is wanting.

If this equation were solved, we could obtain any value of x corresponding to that of u, from the equation

$$x = u + x', \quad \text{since} \quad x = u - \frac{P}{m}.$$

We have, then, in order to make the second term of an equation disappear, the following

RULE.

Substitute for the unknown quantity a new unknown quantity minus the co-efficient of the second term divided by the exponent which expresses the degree of the equation.

Let us apply this rule to the equation,

$$x^2 + 2px = q.$$

If we make $\quad x = u - p,$

we have $\quad (u - p)^2 + 2p(u - p) = q;$

and by performing the indicated operations and transposing, we find

$$u^2 = p^2 + q.$$

263*. Instead of making the second term disappear, it may be required to find an equation which shall be deprived of its third, fourth, or any other term. This is done, by making the co-efficient of u, corresponding to that term, equal to 0.

For example, to make the third term disappear, we make, in the transformed equation, (Art. 263),

$$m\frac{m-1}{2}x'^2 + (m-1)Px' + Q = 0,$$

from which we obtain two values for x', which substituted in the transformed equation, reduce it to the form,

$$u^m + P'u^{m-1} + R'u^{m-3} \ldots + T'u + U' = 0.$$

Beyond the third term it will be necessary to solve an equation of a degree superior to the second, to obtain the value of x'; and to cause the last term to disappear, it will be necessary to solve the equation,

$$x'^m + Px'^{m-1} \dots + Tx' + U = 0,$$

which is what the given equation becomes when x' is substituted for x.

It may happen that the value,

$$x' = -\frac{P}{m},$$

which makes the second term disappear, causes also the disappearance of the third or some other term. For example, in order that the third term may disappear at the same time with the second, it is only necessary that the value of x', which results from the equation,

$$x' = -\frac{P}{m},$$

shall also satisfy the equation,

$$m\frac{m-1}{2}x'^2 + (m-1)Px' + Q = 0.$$

Now, if in this last equation, we replace x' by $-\frac{P}{m}$, we have

$$m\frac{m-1}{2}\frac{P^2}{m^2} - (m-1)\frac{P^2}{m} + Q = 0, \quad \text{or} \quad (m-1)P^2 - 2mQ = 0;$$

and, consequently, if

$$P^2 = \frac{2mQ}{m-1},$$

the disappearance of the second term will also involve that of the third.

Formation of Derived Polynomials.

264. That transformation of an equation which consists in substituting $u + x'$ for x, is of frequent use in the discussion of equations. In practice, there is a very simple method of obtaining the transformed equation which results from this substitution.

To show this, let us substitute for x, $u + x'$ in the equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \dots Tx + U = 0;$$

then, by developing, and arranging the terms according to the ascending powers of u, we have

$$\left.\begin{array}{l|l|l|l}
x'^m & +mx'^{m-1} & u+m\dfrac{m-1}{1.2}x'^{m-2} & u^2+\dots u^m \\
+Px'^{m-1} & +(m-1)Px'^{m-2} & +(m-1)\dfrac{m-2}{1.2}Px'^{m-3} & \\
+Qx'^{m-2} & +(m-2)Qx'^{m-3} & +(m-2)\dfrac{m-3}{1.2}Qx'^{m-4} & \\
+\dots & +\dots & +\dots & \\
+Tx' & +T & & \\
+U & & &
\end{array}\right\} = 0.$$

By examining and comparing the co-efficients of the different powers of u, we see that the co-efficient of u^0, is what the first member of the given equation becomes when x' is substituted in place of x; we shall denote this expression by X'.

The co-efficient of u^1 is formed from the preceding term X', by multiplying each term of X' by the exponent of x' in that term, and then diminishing this exponent by 1; we shall denote this co-efficient by Y'.

The co-efficient of u^2 is formed from Y', by multiplying each term of Y' by the exponent of x' in that term, dividing the product by 2, and then diminishing each exponent by 1. Representing this co-efficient by $\frac{Z'}{2}$, we see that *Z' is formed from Y', in the same manner that Y' is formed from X'.*

In general, the co-efficient of any power of u, in the above transformed equation, may be found from the preceding co-efficient in the following manner, viz.:—

Multiply each term of the preceding co-efficient by the exponent of x' in that term, and diminish the exponent of x' by 1; then divide the algebraic sum of these expressions by the number of preceding co-efficients.

The law by which the co-efficients,

$$X', \quad Y', \quad \frac{Z'}{1.2}, \quad \frac{V'}{1.2.3},$$

are derived from each other, is evidently the same as that which governs the formation of the numerical co-efficients of the terms in the binomial formula.

The expressions, Y', Z', V', W', &c., are called *successive derived polynomials of* X', because each is derived from the preceding one by the same law that Y' is derived from X'.

Generally, any polynomial which is derived from another by the law just explained, is called a *derived polynomial.*

Recollect that X' is what the given polynomial becomes when x' is substituted for x.

Y' is called the *first-derived* polynomial;
Z' is called the *second-derived* polynomial;
V' is called the *third-derived* polynomial;
&c., &c.

We should also remember that, if we make $u = 0$, we shall have $x' = x$, whence X' will become the given polynomial, from which the derived polynomials will then be obtained.

265. Let us now apply the above principles in the following

EXAMPLES.

1. Let it be required to find the derived polynomials of the first member of the equation

$$3x^4 + 6x^3 - 3x^2 + 2x + 1 = 0.$$

Now, u being zero, and $x' = x$, we have from the law of forming the derived polynomials,

$$\begin{aligned}
X' &= 3x^4 + 6x^3 - 3x^2 + 2x + 1; \\
Y' &= 12x^3 + 18x^2 - 6x + 2; \\
Z' &= 36x^2 + 36x - 6; \\
V' &= 72x + 36; \\
W' &= 72.
\end{aligned}$$

It should be remarked that the exponent of x, in the terms 1, 2, -6, 36, and 72, is equal to 0; hence, each of those terms disappears in the following derived polynomial.

2. Let it be required to cause the second term to disappear in the equation

$$x^4 - 12x^3 + 17x^2 - 9x + 7 = 0.$$

Make (Art. 263), $\quad x = u + \frac{12}{4} = u + 3;$

whence, $\quad x' = 3.$

The transformed equation will be of the form

$$X' + Y'u + \frac{Z'}{2}u^2 + \frac{V'}{2 \times 3}u^3 + u^4 = 0,$$

and the operation is reduced to finding the values of the coefficients

$$X', \quad Y', \quad \frac{Z'}{2}, \quad \frac{V'}{2.3}.$$

Now, it follows from the preceding law, for derived polynomials, that

$$X' = (3)^4 - 12.(3)^3 + 17.(3)^2 - 9.(3)^1 + 7, \text{ or } X' = -110;$$

$$Y' = 4.(3)^3 - 36.(3)^2 + 34.(3)^1 - 9, \text{ or } Y' = -123;$$

$$\frac{Z'}{2} = 6.(3)^2 - 36.(3)^1 + 17, \text{ or } \frac{Z'}{2} = -37;$$

$$\frac{V'}{2.3} = 4.(3)^1 - 12 \quad \frac{V'}{2.3} = 0.$$

Therefore, the transformed equation becomes

$$u^4 - 37u^2 - 123u - 110 = 0.$$

3. Transform the equation

$$4x^3 - 5x^2 + 7x - 9 = 0$$

into another equation, the roots of which shall exceed those of the given equation by 1.

Make, $\quad x = u - 1; \quad$ whence $\quad x' = -1;$

and the transformed equation will be of the form

$$X' + Y'u + \frac{Z'}{1.2}u^2 + \frac{V'}{1.2.3}u^3 = 0$$

We have, from the principles established,

$$X' = 4.(-1)^3 - 5.(-1)^2 + 7.(-1)^1 - 9, \text{ or } X' = -25;$$

$$Y' = 12.(-1)^2 - 10.(-1)^1 + 7 \quad \ldots \quad Y' = +29;$$

$$\frac{Z'}{2} = 12.(-1)^1 - 5 \quad \ldots \quad \frac{Z'}{2} = -17;$$

$$\frac{V'}{2.3} = 4 \quad \ldots \quad \frac{V'}{2.3} = +4.$$

Therefore, the transformed equation is,

$$4u^3 - 17u^2 + 29u - 25 = 0.$$

4. What is the transformed equation, if the second term be made to disappear from the equation

$$x^5 - 10x^4 + 7x^3 + 4x - 9 = 0?$$

Ans. $u^5 - 33u^3 - 118u^2 - 152u - 73 = 0.$

5. What is the transformed equation, if the second term be made to disappear from the equation

$$3x^3 + 15x^2 + 25x - 3 = 0?$$

Ans. $u^3 - \dfrac{152}{27} = 0.$

6. Transform the equation

$$3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$$

into another, the roots of which shall be less than the roots of the given equation by $\dfrac{1}{3}$.

Ans. $3u^4 - 9u^3 - 4u^2 - \dfrac{65}{9}\cdot u - \dfrac{34}{3} = 0.$

Properties of Derived Polynomials.

266. We will now develop some of the properties of derived polynomials.

Let $\quad x^m + Px^{m-1} + Qx^{m-2} \ldots Tx + U = 0$

be a given equation, and a, b, c, d, &c., its m roots. We shall then have (Art. 248),

$$x^m + Px^{m-1} + Qx^{m-2} \ldots = (x-a)(x-b)(x-c) \ldots (x-l).$$

Making $$x = x' + u,$$
or omitting the accents, and substituting $x + u$ for x, and we have
$$(x + u)^m + P(x + u)^{m-1} + \ldots = (x + u - a)(x + u - b)\ldots;$$
or, changing the order of x and u, in the second member, and regarding $x - a$, $x - b$, . . . each as a single quantity,
$$(x + u)^m + P(x + u)^{m-1} \ldots = (u + \overline{x - a})(u + \overline{x - b}) \ldots (u + \overline{x - l}).$$

Now, by performing the operations indicated in the two members, we shall, by the preceding article, obtain for the first member,
$$X + Yu + \frac{Z}{2}u^2 + \ldots u^m;$$
X being the first member of the proposed equation, and Y, Z, &c., the derived polynomials of this member.

With respect to the second member, it follows from Art. 251:

1st. That the term involving u^0, or the last term, is equal to the product $(x - a)(x - b) \ldots (x - l)$ of the factors of the proposed equation.

2d. The co-efficient of u is equal to the sum of the products of these m factors, taken $m - 1$ and $m - 1$.

3d. The co-efficient of u^2 is equal to the sum of the products of these m factors, taken $m - 2$ and $m - 2$; and so on.

Moreover, since the two members of the last equation are identical, the co-efficients of the same powers of u in the two members are equal. Hence,
$$X = (x - a)(x - b)(x - c) \ldots (x - l),$$
which was already shown.

Hence, also, Y, or the first derived polynomial, *is equal to the sum of the products of the* m *factors of the first degree in the proposed equation, taken* m − 1 *and* m − 1; *or equal to the algebraic sum of all the quotients that can be obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation, that is,*
$$Y = \frac{X}{x - a} + \frac{X}{x - b} + \frac{X}{x - c} + \ldots \frac{X}{x - l}.$$

Also, $\frac{Z}{2}$, that is, the second derived polynomial, divided by 2, *is equal to the sum of the products of the* m *factors of the first member of the proposed equation, taken* m — 2 *and* m — 2*; or equal to the sum of the quotients obtained by dividing* X *by each of the different factors of the second degree; that is,*

$$\frac{Z}{2} = \frac{X}{(x-a)(x-b)} + \frac{X}{(x-a)(x-c)} \cdots \frac{X}{(x-k)(x-l)}.$$

and so on.

Of Equal Roots.

267. An equation is said to contain equal roots, when its first member contains equal factors of the first degree with respect to the unknown quantity. When this is the case, the derived polynomial, which is the sum of the products of the m factors taken $m-1$ and $m-1$, contains a factor in its different parts, which is two or more times a factor of the first member of the proposed equation (Art. 266): hence,

There must be a common divisor between the first member of the proposed equation, and its first derived polynomial.

It remains to ascertain the relation between this common divisor and the equal factors.

268. *Having given an equation, it is required to discover whether it has equal roots, and to determine these roots if possible.*

Let us make

$$X = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0,$$

and suppose that the second member contains n factors equal to $x-a$, n' factors equal to $x-b$, n'' factors equal to $x-c\ldots$, and also, the simple factors $x-p$, $x-q$, $x-r\ldots$; we shall then have,

$$X = (x-a)^n (x-b)^{n'} (x-c)^{n''} \ldots (x-p)(x-q)(x-r) \quad (1).$$

We have seen that Y, or the derived polynomial of X, is *the sum of the quotients obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation* (Art. 266).

Now, since X contains n factors equal to $x - a$, we shall have n partial quotients equal to $\frac{X}{x-a}$; and the same reasoning applies to each of the repeated factors, $x - b$, $x - c$. . . . . Moreover, we can form but one quotient for each simple factor, which is of the form,

$$\frac{X}{x-p}, \quad \frac{X}{x-q}, \quad \frac{X}{x-r} \cdot \cdot \cdot \cdot$$

therefore, the first derived polynomial is of the form,

$$Y = \frac{nX}{x-a} + \frac{n'X}{x-b} + \frac{n''X}{x-c} + \ldots + \frac{X}{x-p} + \frac{X}{x-q} + \frac{X}{x-r} + \ldots (2)$$

By examining the form of the value of X in equation (1), it is plain that

$$(x-a)^{n-1}, \quad (x-b)^{n'-1}, \quad (x-c)^{n''-1} \ldots$$

are factors common to all the terms of the polynomial Y; hence the product,

$$(x-a)^{n-1} \times (x-b)^{n'-1} \times (x-c)^{n''-1} \ldots$$

is a divisor of Y. Moreover, it is evident that it will also divide X: it is therefore a common divisor of X and Y; and it is their greatest common divisor.

For, the prime factors of X, are $x - a$, $x - b$, $x - c$ . . ., and $x - p$, $x - q$, $x - r$ . . .; now, $x - p$, $x - q$, $x - r$, cannot divide Y, since some one of them will be wanting in some of the parts of Y, while it will be a factor of all the other parts.

Hence, the greatest common divisor of X and Y, is

$$D = (x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1} \; . \; . \; .; \text{ that is,}$$

The greatest common divisor is composed of the product of those factors which enter two or more times in the given equation, each raised to a power less by 1 *than in the primitive equation.*

269. From the above, we deduce the following method for finding the equal roots.

To discover whether an equation,

$$X = 0,$$

contains any equal roots:

1st. *Form Y, or the derived polynomial of X; then seek for the greatest common divisor between X and Y.*

2d. *If one cannot be obtained, the equation has no equal roots, or equal factors.*

If we find a common divisor D, and it is of the first degree, or of the form $x - h$, make $x - h = 0$, whence $x = h$.

We then conclude, that the equation has two roots equal to h, *and has but one species of equal roots, from which it may be freed by dividing X by* $(x - h)^2$.

If D is of the second degree with reference to x, *solve the equation* $D = 0$. There may be two cases; the two roots will be equal, or they will be unequal.

1st. When we find $D = (x - h)^2$, *the equation has three roots equal to* h, *and has but one species of equal roots*, from which it can be freed by dividing X by $(x - h)^3$.

2d. When D is of the form $(x - h)(x - h')$, the proposed equation *has two roots equal to* h, and *two equal to* h′, from which it may be freed by dividing X by $(x - h)^2 (x - h')^2$, or by D^2.

Suppose now that D is of any degree whatever; *it is necessary*, in order to know the species of equal roots, and the number of roots of each species, *to solve completely the equation*,

$$D = 0.$$

Then, every simple root of the equation $D = 0$ will be twice a root of the given equation; every double root of the equation $D = 0$ will be three times a root of the given equation; and so on.

As to the simple roots of

$$X = 0,$$

we begin by freeing this equation of the equal factors contained in it, and the resulting equation, $X' = 0$, will make known the simple roots.

EXAMPLES.

1. Determine whether the equation,

$$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$

contains equal roots.

We have for the first derived polynomial,

$$8x^3 - 36x^2 + 38x - 6.$$

Now, seeking for the greatest common divisor of these polynomials, we find

$$D = x - 3 = 0, \quad \text{whence} \quad x = 3:$$

hence, the given equation has *two* roots equal to 3.

Dividing its first member by $(x-3)^2$, we obtain

$$2x^2 + 1 = 0; \quad \text{whence,} \quad x = \pm \frac{1}{2}\sqrt{-2}.$$

The equation, therefore, is completely solved, and its roots are

$$3, \quad 3, \quad +\frac{1}{2}\sqrt{-2} \ \text{and} \ -\frac{1}{2}\sqrt{-2}.$$

2. For a second example, take

$$x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0.$$

The first derived polynomial is

$$5x^4 - 8x^3 + 9x^2 - 14x + 8;$$

and the common divisor,

$$x^2 - 2x + 1 = (x-1)^2:$$

hence, the proposed equation has *three* roots equal to 1.

Dividing its first member by

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1,$$

the quotient is

$$x^2 + x + 3 = 0; \quad \text{whence,} \quad x = \frac{-1 \pm \sqrt{-11}}{2};$$

thus, the equation is completely solved.

3. For a third example, take the equation

$$x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0.$$

The first derived polynomial is

$$7x^6 + 30x^5 + 30x^4 - 24x^3 - 45x^2 - 6x + 8;$$

and the common divisor is

$$x^4 + 3x^3 + x^2 - 3x - 2.$$

The equation,

$$x^4 + 3x^3 + x^2 - 3x - 2 = 0,$$

cannot be solved directly, but by applying the method of equal roots to it, that is, by seeking for a common divisor between its first member and its derived polynomial,

$$4x^3 + 9x^2 + 2x - 3:$$

we find a common divisor, $x + 1$; which proves that the *square* of $x + 1$ is a factor of

$$x^4 + 3x^3 + x^2 - 3x - 2,$$

and the *cube* of $x + 1$, a factor of the first member of the given equation.

Dividing

$$x^4 + 3x^3 + x^2 - 3x - 2 \quad \text{by} \quad (x + 1)^2 = x^2 + 2x + 1,$$

we have $x^2 + x - 2$, which being placed equal to zero, gives the two roots $x = 1$, $x = -2$, or the two factors, $x - 1$ and $x + 2$. Hence, we have

$$x^4 + 3x^3 + x^2 - 3x - 2 = (x + 1)^2 (x - 1) (x + 2).$$

Therefore, the first member of the proposed equation is equal to

$$(x + 1)^3 (x - 1)^2 (x + 2)^2;$$

that is, the proposed equation has *three* roots equal to -1, *two* equal to $+1$, and *two* equal to -2.

4. What is the product of the equal factors of the equation

$$x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 48x + 36 = 0?$$

Ans. $(x - 2)^2 (x - 3)^2 (x + 1)^3$.

5. What is the product of the equal factors in the equation,

$$x^7 - 3x^6 + 9x^5 - 19x^4 + 27x^3 - 33x^2 + 27x - 9 = 0?$$

Ans. $(x - 1)^3 (x^2 + 3)^2$.

Elimination.

270. We have already explained the methods of eliminating one unknown quantity from two equations, when these equations are of the first degree with respect to the unknown quantities. When the equations are of a higher degree than the first, the methods explained are not in general applicable. In this case, *the method of the greatest common divisor* is considered the best, and it is this method that we now propose to investigate.

One quantity is said to be a *function* of another when it depends upon that other for its value; that is, when the quantities are so connected, that the value of the latter cannot be changed without producing a corresponding change in the former.

271. If two equations, containing two unknown quantities, be combined, so as to produce a single equation containing but one unknown quantity, the resulting equation is called a *final equation;* and the roots of this equation are called *compatible values* of the unknown quantity which enters it.

Let us assume the equations,

$$P = 0 \quad \text{and} \quad Q = 0,$$

in which P and Q are functions of x and y of any degree whatever; it is required to combine these equations in such a manner as to eliminate one of the unknown quantities.

If we suppose the final equation involving y to be found, and that $y = a$ is a root of this equation, it is plain that this value of y, in connection with some value of x, will satisfy both equations.

If then, we substitute this value of y in both equations, there will result two equations containing only x, and these equations will have at least one root in common, and consequently, their first members will have a common divisor involving x (Art. 246).

This common divisor will be of the first, or of a higher degree with respect to x, according as the particular value of $y = a$ corresponds to one or more values of x.

Conversely, *every value of y which*, being substituted in the two equations, gives a common divisor involving x, *is necessarily a compatible value*, for it then satisfies the two equations at the same time with the value or values of x found from this common divisor when put equal to 0.

272. We will remark, that, before *the substitution*, *the first members of the equations cannot*, in general, *have a common divisor* which is a function of one or both of the unknown quantities.

For, let us suppose, for a moment, that the equations

$$P = 0 \quad \text{and} \quad Q = 0,$$

are of the form

$$P' \times R = 0 \quad \text{and} \quad Q' \times R = 0,$$

R being a function of both x and y.

Placing $R = 0$, we obtain a single equation involving two unknown quantities, which can be satisfied with an *infinite number of systems of values.* Moreover, every system which renders R equal to 0, would at the same time cause $P'.R$ and $Q'.R$ to become 0, and consequently, would satisfy the equations

$$P = 0 \quad \text{and} \quad Q = 0.$$

Thus, the hypothesis of a common divisor of the two polynomials P and Q, containing x and y, brings with it, as a consequence, that the proposed equations are indeterminate. Therefore, if there exists a common divisor, involving x and y, of the two polynomials P and Q, the proposed equations will be *indeterminate*, that is, they may be satisfied by an infinite number of systems of values of x and y. Then there is no data to determine a *final equation* in y, since the number of values of y is *infinite.*

Again, let us suppose that R is a function of x only.

Placing $R = 0$, we shall, if the equation be solved with reference to x, obtain one or more values for this unknown quantity.

Each of these values, substituted in the equations

$$P'.R = 0 \quad \text{and} \quad Q'.R = 0,$$

will satisfy them, whatever value we may attribute to y, since these values of x would reduce R to 0, independently of y. Therefore, in this case, the proposed equations admit of a *finite number of values* for x, but of an infinite number of values for y and then, therefore, there cannot exist a final equation in y.

Hence, when the equations

$$P = 0, \qquad Q = 0$$

are determinate, that is, when they admit only of a *limited number* of systems of values for x and y, their first members cannot have *for a common divisor a function of these unknown quantities*, unless a particular substitution has been made for one of these quantities.

273. From this it is easy to deduce a process for obtaining the *final equation* involving y.

Since the characteristic property of every compatible value of y is, that being substituted in the first members of the two equations, it gives them a common divisor involving x, which they had not before, it follows, that if to the two proposed polynomials, arranged with reference to x, we apply the process for finding the greatest common divisor, we shall generally not find one. But, by continuing the operation properly, we shall arrive at a remainder independent of x, but which is a function of y, and which, placed equal to 0, will give the required *final equation*.

For, every value of y found from this equation, reduces to zero the last remainder in the operation for finding the common divisor; it is, then, such that being substituted in the preceding remainder, it will render this remainder a common divisor of the first members P and Q. Therefore, each of the roots of the equation thus formed, is a compatible value of y.

274. Admitting that the final equation may be completely solved, which would give all the compatible values, it would afterward be necessary to obtain the corresponding values of x. Now, it is evident that it would be sufficient for this, to substitute the different values of y in the remainder preceding the

last, put the polynomial involving x which results from it, equal to 0, and find from it the values of x; for these polynomials are nothing more than the divisors involving x, which become common to A and B.

But as the final equation is generally of a degree superior to the second, we cannot here explain the methods of finding the values of y. Indeed, our design was principally to show that, *two equations of any degree being given, we can, without supposing the resolution of any equation, arrive at another equation, containing only one of the unknown quantities which enter into the proposed equations.*

EXAMPLES.

1. Having given the equations

$$x^2 + xy + y^2 - 1 = 0,$$
$$x^3 + y^3 = 0,$$

to find the final equation in y.

First Operation.

$$\begin{array}{l|l}
x^3 + y^3 & x^2 + xy + y^2 - 1 \\ \cline{2-2}
x^3 + yx^2 + (y^2 - 1)x & x - y \\ \cline{1-1}
-yx^2 - (y^2 - 1)x + y^3 & \\
-yx^2 - y^2x - y^3 + y & \\ \cline{1-1}
x + 2y^3 - y = \text{1st remainder.} &
\end{array}$$

Second Operation.

$$\begin{array}{l|l}
x^2 + yx + y^2 - 1 & x + 2y^3 - y \\ \cline{2-2}
x^2 + (2y^3 - y)x & x - (2y^3 - 2y) \\ \cline{1-1}
-(2y^3 - 2y)x + y^2 - 1 & \\
-(2y^3 - 2y)x - 4y^6 + 6y^4 - 2y^2 & \\ \cline{1-1}
4y^6 - 6y^4 + 3y^2 - 1. &
\end{array}$$

Hence, the final equation in y, is

$$4y^6 - 6y^4 + 3y^2 - 1 = 0.$$

If it were required to find the final equation in x, we observe that x and y enter into the primitive equations under the same forms; hence, x may be changed into y and y into x, without destroying the equality of the members. Therefore,

$$4x^6 - 6x^4 + 3x^2 - 1 = 0$$

is the final equation in x.

2. Find the final equation in y, from the equations

$$x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0,$$
$$x^2 - 2yx + \quad y^2 - y = 0.$$

First Operation.

$$\begin{array}{l|l}
x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y & x^2 - 2xy + y^2 - y \\ \hline
x^3 - 2yx^2 + \quad (y^2 - y)x & x - y \\ \hline
\quad - yx^2 + (2y^2 + 1)x - y^3 + y^2 - 2y & \\
\quad - yx^2 + \ 2y^2x \qquad - y^3 + y^2 & \\ \hline
\qquad\qquad x - 2y &
\end{array}$$

Second Operation.

$$\begin{array}{l|l}
x^2 - 2xy + y^2 - y & x - 2y \\ \hline
x^2 - 2xy & x \\ \hline
\qquad y^2 - y. &
\end{array}$$

Hence, $$y^2 - y = 0,$$

is the final equation in y. This equation gives

$$y = 1 \quad \text{and} \quad y = 0.$$

Placing the preceding remainder equal to zero, and substituting therein the values of y,

$$y = 1 \quad \text{and} \quad y = 0,$$

we find for the corresponding values of x,

$$x = 2 \quad \text{and} \quad x = 0;$$

from which the given equations may be entirely solved.

CHAPTER XI.

SOLUTION OF NUMERICAL EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.—STURM'S THEOREM.—CARDAN'S RULE.—HORNER'S METHOD.

275. The principles established in the preceding chapter, are applicable to all equations, whether the co-efficients are numerical or algebraic. These principles are the elements which are employed in the solution of all equations of higher degrees.

Algebraists have hitherto been unable to solve equations of a higher degree than the fourth. The formulas which have been deduced for the solution of algebraic equations of the higher degrees, are so complicated and inconvenient, even when they can be applied, that we may regard the general solution of an algebraic equation, of any degree whatever, as a problem more curious than useful.

Methods have, however, been found for determining, to any degree of exactness, the values of the roots of all numerical equations; that is, of those equations which, besides the unknown quantity, involve only numbers.

It is proposed to develop these methods in this chapter.

276. To render the reasoning general, we will take the equation,

$$X = x^m + Px^{m-1} + Qx^{m-2} + \ldots U = 0.$$

in which P, Q . . . denote particular numbers which are real, and either positive or negative.

If we substitute for x a number a, and denote by A what X becomes under this supposition; and again substitute $a + u$ for x, and denote the new polynomial by A': then, *u may be taken so small, that the difference between A' and A shall be less than any assignable quantity.*

If, now, we denote by B, C, D, ... what the co-efficients Y, $\frac{Z}{2}$, $\frac{V}{2\,.\,3}$ (Art. 264), become, when we make $x = a$, we shall have,

$$A' = A + Bu + Cu^2 + Du^3 + \ldots + u^m \quad - - - \quad (1);$$

whence,

$$A' - A = Bu + Cu^2 + Du^3 + \ldots + u^m \quad - - - \quad (2).$$

It is now required to show that this difference may be rendered less than any assignable quantity, by attributing a value sufficiently small to u.

If it be required to make the difference between A' and A less than the number N, we must assign a value to u which will satisfy the inequality

$$Bu + Cu^2 + Du^3 + \ldots . u^m < N \quad - - - \quad (3).$$

Let us take the most unfavorable case that can occur, viz., let us suppose that every co-efficient is positive, and that each is equal to the largest, which we will designate by K. Then any value of u which will satisfy the inequality

$$K(u + u^2 + u^3 + \ldots . u^m) < N \quad - - - \quad (4),$$

will evidently satisfy inequality (3).

Now, the expression within the parenthesis is a geometrical progression, whose first term is u, whose last term is u^m, and whose ratio is u; hence (Art. 188),

$$u + u^2 + u^3 + \ldots u^m = \frac{u^{m+1} - u}{u - 1} = \frac{u - u^{m+1}}{1 - u} = \frac{u}{1 - u} \times (1 - u^m).$$

Substituting this value in inequality (4), we have,

$$\frac{Ku}{1 - u}(1 - u^m) < N \quad - - - - \quad (5).$$

If now we make $u = \frac{N}{N + K}$, the first factor of the first member of inequality (5) reduces to N and since $\left(\frac{N}{N + K}\right)^m$ is less than 1, the second factor is less than 1; hence, the first member is less than N.

We conclude, therefore, that $u = \frac{N}{N+K}$, and every smaller value of u, will satisfy the inequalities (3) and (4), and consequently, make the difference between A' and A less than any assignable number N.

If in the value of A', equation (1), we make $u = \frac{A}{A+K}$, it is plain that the sum of the terms

$$Bu + Cu^2 + Du^3 + \dots u^m$$

will be less than A, from what has just been proved; whence we conclude that

In a series of terms arranged according to the ascending powers of an arbitrary quantity, a value may be assigned to that so small, as to make the first term numerically greater than the sum of all the other terms.

First Principle.

277. *If two numbers* p *and* q, *substituted in succession in the place of* x *in the first member of a numerical equation, give results affected with contrary signs, the proposed equation has a real root, comprehended between these two numbers.*

Let us suppose that p, when substituted for x in the first member of the equation

$$X = 0, \quad \text{gives} \quad +R,$$

and that q, substituted in the first member of the equation

$$X = 0, \quad \text{gives} \quad -R'.$$

Let us now suppose x to vary between the values of p and q by so small a quantity, that the difference between any two corresponding consecutive values of X shall be less than any assignable quantity (Art. 276), in which case, we say that X is subject to the *law of continuity*, or that it passes through all the intermediate values between R and $-R'$.

Now, a quantity which is constantly finite, and subject to the law of continuity, cannot change its sign from positive to nega

tive, or from negative to positive, without passing through zero: hence, there is at least one number between p and q which will satisfy the equation

$$X = 0,$$

and consequently, one root of the equation lies between these numbers.

278. We have shown in the last article, that if two numbers be substituted, in succession, for the unknown quantity in any equation, and give results affected with contrary signs, that there will be at least one real root comprehended between them. We are not, however, to conclude that there may not be more than one; nor are we to infer the converse of the proposition, viz., that the substitution, in succession, of two numbers which *include* roots of the equation, will *necessarily* give results affected with contrary signs.

Second Principle.

279. *When an uneven number of the real roots of an equation is comprehended between two numbers, the results obtained by substituting these numbers in succession for* x *in the first member, will have contrary signs; but if they comprehend an even number of roots, the results obtained by their substitution will have the same sign.*

To make this proposition as clear as possible, denote by a, b, c, . . . those roots of the proposed equation,

$$X = 0,$$

which are supposed to be comprehended between p and q, and by Y, the product of the factors of the first degree, with reference to x, corresponding to the remaining roots of the given equation.

The first member, X, can then be put under the form

$$(x - a)(x - b)(x - c) \ . \ . \ . \times Y = 0.$$

Now, substituting p and q in place of x, in the first member, we shall obtain the two results,

$$(p - a)(p - b)(p - c) \ . \ . \ . \times Y',$$

$$(q - a)(q - b)(q - c) \ . \ . \ . \times Y''.$$

Y' and Y'' representing what Y becomes, when we replace in succession, x by p and q. These two quantities Y' and Y'', are affected with the same sign; for, if they were not, by the first principle there would be at least one other real root comprised between p and q, which is contrary to the hypothesis.

To determine the signs of the above results more easily, divide the first by the second, and we obtain

$$\frac{(p-a)(p-b)(p-c)\ldots\times Y'}{(q-a)(q-b)(q-c)\ldots\times Y''},$$

which can be written thus,

$$\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\ldots\frac{Y'}{Y''}.$$

Now, since the root a is comprised between p and q, that is, is greater than one and less than the other, $p-a$ and $q-a$ must have contrary signs; also, $p-b$ and $q-b$ must have contrary signs, and so on.

Hence, the quotients

$$\frac{p-a}{q-a},\quad \frac{p-b}{q-b},\quad \frac{p-c}{q-c},\quad \text{\&c.},$$

are all *negative*.

Moreover, $\dfrac{Y'}{Y''}$ is essentially positive, since Y' and Y'' are affected with the same sign; therefore, the product

$$\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\ldots\frac{Y'}{Y''},$$

will be *negative*, when the number of roots, $a, b, c \ldots$, comprehended between p and q, is uneven, and *positive* when the number is even.

Consequently, the two results,

$$(p-a)(p-b)(p-c)\ldots\times Y',$$

and

$$(q-a)(q-b)(q-c)\ldots\times Y'';$$

will have contrary signs when the number of roots comprised between p and q is *uneven*, and the same sign when the number is *even*.

Third Principle.

280. *If the signs of the alternate terms of an equation be changed, the signs of the roots will be changed.*

Take the equation,

$$x^m + Px^{m-1} + Qx^{m-2} \ldots + U = 0 \quad - \quad - \quad (1);$$

and by changing the signs of the alternate terms, we have

$$x^m - Px^{m-1} + Qx^{m-2} \ldots \pm U = 0 \quad - \quad - \quad (2),$$

or,

$$-x^m + Px^{m-1} - Qx^{m-2} \ldots \mp U = 0 \quad - \quad - \quad (3).$$

But equations (2) and (3) are the same, since the sum of the positive terms of the one is equal to the sum of the negative terms of the other, whatever be the value of x.

Suppose a to be a root of equation (1); then, the substitution of a for x will verify that equation. But the substitution of $-a$ for x, in either equations (2) or (3), will give the same result as the substitution of $+a$, in equation (1): hence $-a$, is a root of equation (2), or of equation (3).

We may also conclude, that if the signs of all the terms be changed, the signs of the roots will not be altered.

Limits of Real Roots.

281. The different methods for resolving numerical equations, consist, generally, in substituting particular numbers in the proposed equation, in order to discover if these numbers verify it, or whether there are roots comprised between them. But by reflecting a little on the composition of the first member of the general equation,

$$x^m + Px^{m-1} + Qx^{m-2} \ldots + Tx + U = 0,$$

we become sensible, that there are certain numbers, above which it would be useless to substitute, because all numbers above a certain limit would give positive results.

282. It is now required *to determine a number, which being substituted for* x *in the general equation, will render the first term* x^m *greater than the arithmetical sum of all the other terms*; that is, it is required to find a number for x which will render

$$x^m > Px^{m-1} + Qx^{m-2} + \ldots + Tx + U.$$

Let k denote the greatest numerical co-efficient, and substitute it in place of each of the co-efficients; the inequality will then become

$$x^m > kx^{m-1} + kx^{m-2} + \ldots + kx + k.$$

It is evident that every number substituted for x which will satisfy this condition, will satisfy the preceding one. Now, dividing both members of this inequality by x^m, it becomes

$$1 > \frac{k}{x} + \frac{k}{x^2} + \frac{k}{x^3} + \ldots + \frac{k}{x^{m-1}} + \frac{k}{x^m}.$$

Making $x = k$, the second member reduces to 1 plus the sum of several fractions. The number k will not therefore satisfy the inequality; but if we make $x = k + 1$, we obtain for the second member the expression,

$$\frac{k}{k+1} + \frac{k}{(k+1)^2} + \frac{k}{(k+1)^3} + \ldots + \frac{k}{(k+1)^{m-1}} + \frac{k}{(k+1)^m}.$$

This is a geometrical progression, the first term of which is $\frac{k}{k+1}$, the last term, $\frac{k}{(k+1)^m}$, and the ratio, $\frac{1}{k+1}$; hence, the expression reduces to

$$\frac{\dfrac{k}{(k+1)^{m+1}} - \dfrac{k}{k+1}}{\dfrac{1}{k+1} - 1} = 1 - \frac{1}{(k+1)^m},$$

which is evidently less than 1.

Now, any number $> (k + 1)$, put in place of x, will render the sum of the fractions $\frac{k}{x} + \frac{k}{x^2} + \ldots$ still less: therefore,

The greatest co-efficient plus 1, *or any greater number, being substituted for* x, *will render the first term* x^m *greater than the arithmetical sum of all the other terms.*

283. Every number which exceeds the greatest of the positive roots of an equation, *is called a superior limit of the positive roots.*

From this definition, it follows, that this limit is susceptible of an infinite number of values. For, when a number is found to exceed the greatest positive root, every number greater than this, is also a superior limit. The term, however, is generally applied to that value nearest the value of the root.

Since the greatest of the positive roots will, when substituted for x, merely reduce the first member to zero, it follows, that we shall be sure of obtaining a superior limit of the positive roots by finding *a number, which substituted in place of* x, *renders the first member positive, and which at the same time is such, that every greater number will also give a positive result;* hence,

The greatest co-efficient of x *plus* 1, *is a superior limit of the positive roots.*

Ordinary Limit of the Positive Roots.

284. The limit of the positive roots obtained in the last article, is commonly much too great, because, in general, the equation contains several positive terms. We will, therefore, seek for a limit suitable to all equations.

Let x^{m-n} denote that power of x that enters the first negative term which follows x^m, and let us consider the most unfavorable case, viz., that in which all the succeeding terms are negative, and the co-efficient of each is equal to the greatest of the negative co-efficients in the equation.

Let S denote this co-efficient. What conditions will render

$$x^m > Sx^{m-n} + Sx^{m-n-1} + \ldots Sx + S\,?$$

Dividing both members of this inequality by x^m, we have

$$1 > \frac{S}{x^n} + \frac{S}{x^{n+1}} + \frac{S}{x^{n+2}} + \ldots + \frac{S}{x^{m-1}} + \frac{S}{x^m}.$$

Now, by supposing

$$x = \sqrt[n]{S} + 1, \quad \text{or for simplicity, making } \sqrt[n]{S} = S'.$$

which gives, $\quad S = S'^n, \quad$ and $\quad x = S' + 1,$

the second member of the inequality will become,

$$\frac{S'^n}{(S'+1)^n}+\frac{S'^n}{(S'+1)^{n+1}}+\ldots+\frac{S'^n}{(S'+1)^{m-1}}+\frac{S'^n}{(S'+1)^m},$$

which is a geometrical progression, of which $\frac{S'^n}{(S'+1)^n}$ is the first term, and $\frac{1}{S'+1}$ the ratio. Hence, the expression for the sum of all the terms is (Art. 188),

$$\frac{\dfrac{S'^n}{(S'+1)^{m+1}}-\dfrac{S'^n}{(S'+1)^n}}{\dfrac{1}{S'+1}\quad -1}=\frac{S'^{n-1}}{(S'+1)^{n-1}}-\frac{S'^{n-1}}{(S'+1)^m}<1$$

Moreover, every number $> S'+1$ or $\sqrt[n]{S}+1$, will, when substituted for x, render the sum of the fractions

$$\frac{S}{x^n}+\frac{S}{x^{n+1}}+\ldots$$

still smaller, since the numerators remain the same, while the denominators are increased. Hence, this sum will also be less.

Hence, $\sqrt[n]{S}+1$, and every greater number, being substituted for x, will render the first term x^m greater than the arithmetical sum of all the negative terms of the equation, and will consequently give a positive result for the first member. Therefore,

That root of the numerical value of the greatest negative co-efficient whose index is equal to the number of terms which precede the first negative term, increased by 1, *is a superior limit of the positive roots of the equation. If the co-efficient of a term is* 0, *the term must still be counted.*

Make $n=1$, in which case the first negative term is the second term of the equation; the limit becomes

$$\sqrt[1]{S}+1=S+1;$$

that is, *the greatest negative co-efficient plus* 1.

Let $n=2$; then, the limit is $\sqrt[2]{S}+1$. When $n=3$, the limit is $\sqrt[3]{S}+1$.

EXAMPLES.

1. What is the superior limit of the positive roots of the equation

$$x^4 - 5x^3 + 37x^2 - 3x + 39 = 0?$$

$$Ans. \ \sqrt[n]{S} + 1 = \sqrt[1]{5} + 1 = 6.$$

2. What is the superior limit of the positive roots of the equation

$$x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0?$$

$$Ans. \ \sqrt[n]{S} + 1 = \sqrt{49} + 1 = 8.$$

3. What is the superior limit of the positive roots of the equation

$$x^4 + 11x^2 + 25x - 67 = 0?$$

In this example, we see that the second term is wanting, that is, its co-efficient is zero; but the term must still be counted in fixing the value of n. We also see, that the largest negative co-efficient of x is found in the last term, where the exponent of x is zero. Hence,

$$\sqrt[n]{S} + 1 = \sqrt[3]{67} + 1;$$

and therefore, 6 is the least whole number that will certainly fulfil the conditions.

Smallest Limit in Entire Numbers.

285. In Art. 282, it was shown that the greatest co-efficient of x plus 1, is a superior limit of the positive roots. In the last article we found a limit still less; and we now propose to find the smallest limit, in whole numbers.

Let

$$X = 0$$

be the proposed equation. If in this equation we make $x = x' + u$, x' being arbitrary, we shall obtain (Art. 264),

$$X' + Y'u + \frac{Z'}{2}u^2 + \ldots + u^m = 0, \quad (1).$$

Let us suppose, that after successive trials we have determined a number for x', which substituted in

$$X', \quad Y', \quad \frac{Z'}{2}$$

renders, at the same time, all these co-efficients positive, this number will in general be greater than the greatest positive root of the equation

$$X = 0.$$

For, if the co-efficients of equation (1) are all positive, no positive value of u can satisfy it; therefore, *all* the real values of u must be negative. But from the equation

$$x = x' + u, \text{ we have } u = x - x';$$

and in order that every value of u, corresponding to each of the values of x and x', may be negative, it is necessary that the greatest positive value of x should be less than the value of x'.

Hence, this value of x' is a superior limit of the positive roots. If we now substitute in succession for x in X the values $x' - 1$, $x' - 2$, $x' - 3$, &c., until a value is found which will make X negative, then the last number which rendered it positive will be the least superior limit of the positive roots in whole numbers.

EXAMPLE.

Let $\qquad x^4 - 5x^3 - 6x^2 - 19x + 7 = 0.$

As x' is indeterminate, we may, to avoid the inconvenience of writing the primes, retain the letter x, in the formation of the derived polynomials; and we have,

$$X = x^4 - 5x^3 - 6x^2 - 19x + 7,$$

$$Y = 4x^3 - 15x^2 - 12x - 19,$$

$$\frac{Z}{2} = 6x^2 - 15x - 6,$$

$$\frac{V}{2.3} = 4x - 5.$$

The question is now reduced to finding the smallest entire number which, substituted in place of x, will render all of these polynomials positive.

It is plain that 2 and every number > 2, will render the polynomial of the first degree positive.

But 2, substituted in the polynomial of the second degree, gives a negative result; and 3, or any number > 3, gives a positive result.

Now, 3 and 4, substituted in succession in the polynomial of the third degree, give negative results; but 5, and any greater number, gives a positive result.

Lastly, 5 substituted in X, gives a negative result, and so does 6; for the first three terms, $x^4 - 5x^3 - 6x^2$, are equivalent to the expression $x^3(x-5) - 6x^2$, which reduces to 0 when $x = 6$; but $x = 7$ evidently gives a positive result. Hence 7, is the least limit in entire numbers. We see that 7 is a superior limit, and that 6 is not; hence, 7 is the least limit, as above shown.

2. Applying this method to the equation,

$$x^5 - 3x^4 - 8x^3 - 25x^2 + 4x - 39 = 0,$$

the superior limit is found to be 6.

3. We find 7 to be the superior limit of the positive roots of the equation,

$$x^5 - 5x^4 - 13x^3 + 17x^2 - 69 = 0.$$

This method is seldom used, except in finding incommensurable roots.

Superior Limit of Negative Roots.—Inferior Limit of Positive and Negative Roots.

286. Having found the superior limit of the positive roots, it remains to find the inferior limit, and the superior and inferior limits of the negative roots, numerically considered.

First, If, in any equation,

$$X = 0, \quad \text{we make} \quad x = \frac{1}{y},$$

we shall have a new equation $Y = 0$.

Since we know, from the relation $x = \dfrac{1}{y}$, that the greatest

positive value of y in the new equation corresponds to the least positive value of x in the given equation, it follows, that

If we determine the superior limit of the positive roots of the equation $Y = 0$, *its reciprocal will be the inferior limit of the positive roots of the given equation.*

Hence, if we designate the superior limit of the positive roots of the equation $Y = 0$ by L', we shall have for the inferior limit of the positive roots of the given equation, $\frac{1}{L'}$.

Second, If in the equation

$$X = 0, \quad \text{we make} \quad x = -y,$$

which gives the transformed equation $Y' = 0$, it is clear that the positive roots of this new equation, taken with the sign $-$, will give the negative roots of the given equation; therefore, determining by known methods, the superior limit of the positive roots of the new equation $Y' = 0$, and designating this limit by L'', we shall have $-L''$ for the superior limit, (numerically), of the negative roots of the given equation.

Third, If in the equation

$$X = 0, \quad \text{we make} \quad x = -\frac{1}{y},$$

we shall have the derived equation $Y'' = 0$. The greatest positive value of y in this equation will correspond to the least negative value (numerically) of x in the given equation. If, then, we find the superior limit of the positive roots of the equation $Y'' = 0$, and designate it by L''', we shall have the inferior limit of the negative roots (numerically) equal to $-\frac{1}{L'''}$.

Consequences deduced from the preceding Principles.

First.

287. *Every equation in which there are no variations in the signs*, that is, *in which all the terms are positive, must have all of its real roots negative;* for, every positive number substituted for x, will render the first member essentially positive.

Second.

288. *Every complete equation*, having its terms alternately positive and negative, must have its real roots all positive; for, every negative number substituted for x in the proposed equation, would render all the terms positive, if the equation be of an even degree, and all of them negative, if it be of an odd degree. Hence, their sum could not be equal to zero in either case.

This principle is also true for *every incomplete equation, in which there results, by substituting* $-y$ *for* x, *an equation having all its terms affected with the same sign.*

Third.

289. Every equation of an odd degree, the co-efficients of which are real, *has at least one real root affected with a sign contrary to that of its last term.*

For, let

$$x^m + Px^{m-1} + \ldots Tx \pm U = 0,$$

be the proposed equation; and first consider the case in which the last term is *negative.*

By making $x = 0$, the first member becomes $-U$. But by giving a value to x equal to the greatest co-efficient plus 1, or $(K+1)$, the first term x^m will become greater than the arithmetical sum of all the others (Art. 282), the result of this substitution will therefore be *positive;* hence, *there is at least one real root comprehended between* 0 *and* $K+1$, which root is positive, and consequently affected with a sign *contrary* to that of the last term (277).

Suppose now, that the last term is *positive.*

Making $x = 0$ in the first member, we obtain $+U$ for the result; but by putting $-(K+1)$ in place of x, we shall obtain a *negative* result, since the first term becomes negative by this substitution; hence, the equation has at least one real root comprehended between 0 and $-(K+1)$, which is negative, or *affected with a sign contrary* to that of the last term.

Fourth.

290. *Every equation of an even degree,* which involves only rea. co-efficients, *and of which the last term is negative, has at least two real roots, one positive and the other negative.*

For, let $-U$ be the last term; making $x=0$, there results $-U$. Now, substitute either $K+1$, or $-(K+1)$, K being the greatest co-efficient in the equation. As m is an even number, the first term x^m will remain positive; besides, by these substitutions, it becomes greater than the sum of all the others; therefore, the results obtained by these substitutions are both *positive*, or affected with a sign contrary to that given by the hypothesis $x=0$; hence, the equation *has at least two real roots*, one *positive*, and comprehended between 0 and $K+1$, the other *negative*, and comprehended between 0 and $-(K+1)$ (277).

Fifth.

291. *If an equation, involving only real co-efficients, contains imaginary roots, the number of such roots must be even.*

For, conceive that the first member has been divided by all the simple factors corresponding to the real roots; the co-efficients of the *quotient* will be real (Art. 246); *and the quotient must alsc be of an even degree;* for, if it was uneven, by placing it equal to zero, we should obtain an equation that would contain at least one real root (289); hence, the imaginary roots must enter by pairs.

Remark.—There is a property of the above polynomial quotient which belongs exclusively to equations containing only imaginary roots; viz., *every such equation always remains positive for any real value substituted for* x.

For, by substituting for x, $K+1$, the greatest co-efficient plus 1, we could always obtain a positive result; hence, if the polynomial could become negative, it would follow that when placed equal to zero, there would be at least one real root com-

prehended between $K+1$ and the number which would give a negative result (Art. 277).

It also follows, that the last term of this polynomial must be *positive*, otherwise $x=0$ would give a negative result.

Sixth.

292. *When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative, the number of such roots is uneven.*

For, first suppose that the last term is $+U$, or *positive*. Since by making $x=0$, there will result $+U$, and by making $x=K+1$, the result will also be positive, it follows that 0 and $K+1$ give two results affected with the same sign, and consequently (Art. 279), the number of real roots, if any, comprehended between them, is even.

When the last term is $-U$, then 0 and $K+1$ give two results affected with contrary signs, and consequently, they comprehend either a *single root*, or *an odd number of them.*

The converse of this proposition is evidently true.

Descartes' Rule.

293. *An equation of any degree whatever, cannot have a greater number of positive roots than there are* variations *in the signs of its terms, nor a greater number of negative roots than there are* permanences *of these signs.*

A *variation* is a change of sign in passing along the terms. A *permanence* is when two consecutive terms have the same sign.

In the equation

$$x - a = 0,$$

there is one variation, and one positive root, $x=a$.

And in the equation $x+b=0$, there is one permanence, and one negative root, $x=-b$.

If these equations be multiplied together, member by member, there will result an equation of the second degree,

$$\left.\begin{array}{c|c} x^2 - a & x - ab \\ + b & \end{array}\right\} = 0.$$

If a is less than b, the equation will be of the first form (Art. 117); and if $a > b$, the equation will be of the second form; that is,

$$a < b \quad \text{gives} \quad x^2 + 2px - q = 0,$$

and

$$a > b \quad \text{"} \quad x^2 - 2px - q = 0.$$

In the first case, there is one permanence and one variation, and in the second, one variation and one permanence. Since in either form, one root is positive and one negative, it follows that there are as many positive roots as there are variations, and as many negative roots as there are permanences.

The proposition will evidently be demonstrated in a general manner, if it be shown that the multiplication of the first member of any equation by a factor $x - a$, corresponding to a *positive* root, introduces *at least one variation*, and that the multiplication by a factor $x + a$, corresponding to a negative root, introduces *at least one permanence.*

Take the equation,

$$x^m \pm Ax^{m-1} \pm Bx^{m-2} \pm Cx^{m-3} \pm \ . \ . \ . \ \pm Tx \pm U = 0,$$

in which the signs succeed each other in any manner whatever. By multiplying by $x - a$, we have

$$\left.\begin{array}{l|l|l|l|l} x^{m+1} \pm A & x^m \pm B & x^{m-1} \pm C & x^{m-2} \pm \ . \ . \ . \ \pm U & x \\ \quad\;\; - a & \quad\;\; \mp Aa & \quad\;\; \mp Ba & \qquad\qquad\qquad \mp Ta & \;\; \mp Ua \end{array}\right\} = 0.$$

The co-efficients which form the first horizontal line of this product, are those of the given equation, taken with the same signs; and the co-efficients of the second line are formed from those of the first, by multiplying by a, changing the signs, and advancing each one place to the right.

Now, so long as each co-efficient in the upper line is greater than the corresponding one in the lower, it will determine the sign of the total co-efficient; hence, in this case there will be, from the first term to that preceding the last, inclusively, the same variations and the same permanences as in the proposed equation; but the last term $\mp Ua$ having a sign contrary to that which immediately precedes it, there must be *one more* variation than in the proposed equation.

When a co-efficient in the lower line is affected with a sign contrary to the one corresponding to it in the upper, and is also greater than this last, there is a change from a permanence of sign to a variation; for the sign of the term in which this happens, being the same as that of the inferior co-efficient, must be contrary to that of the preceding term, which has been supposed to be the same as that of the superior co-efficient. Hence, each time we descend from the upper to the lower line, in order to determine the sign, there is a variation which is not found in the proposed equation; and if, after passing into the lower line, we continue in it throughout, we shall find for the remaining terms the same variations and the same permanences as in the given equation, since the co-efficients of this line are all affected with signs contrary to those of the primitive co-efficients. This supposition would therefore give us one variation for each positive root. But if we ascend from the lower to the upper line, there may be either a variation or a permanence. But even by supposing that this passage produces permanences in all cases, since the last term $\mp Ua$ forms a part of the lower line, it will be necessary to go once more from the upper line to the lower, than from the lower to the upper. Hence, the new equation *must have at least one more variation than the proposed;* and it will be the same for each positive root introduced into it.

It may be demonstrated, in an analogous manner, that *the multiplication of the first member by a factor* $x + a$, corresponding to a negative root, *would introduce one permanence more.* Hence, in any equation, the number of positive roots cannot be greater than the number of VARIATIONS of signs, nor the number of negative roots greater than the number of PERMANENCES.

Consequence.

294. When the roots of an equation are all real, *the number of positive roots is equal to the number of variations, and the number of negative roots to the number of permanences.*

For, let m denote the degree of the equation, n the number of variations of the signs, p the number of permanences; then,

$$m = n + p.$$

Moreover, let n' denote the number of positive roots, and p' the number of negative roots, we shall have

$$m = n' + p';$$

whence, $\quad n + p = n' + p', \quad$ or, $\quad n - n' = p' - p.$

Now, we have just seen that n' cannot be $> n$, nor can it be less, since p' cannot be $> p$; therefore, we must have

$$n' = n, \quad \text{and} \quad p' = p.$$

REMARK.—When an equation wants some of its terms, we can often discover the presence of imaginary roots, by means of the above rule.

For example, take the equation

$$x^3 + px + q = 0,$$

p and q being essentially positive; introducing the term which is wanting, by affecting it with the coefficient ± 0; it becomes

$$x^3 \pm 0 \cdot x^2 + px + q = 0.$$

By considering only the superior sign, we should obtain only permanences, whereas the inferior sign gives two variations. This proves that the equation has some imaginary roots; for, if they were all three real, it would be necessary, by virtue of the superior sign, that they should be all negative, and, by virtue of the inferior sign, that two of them should be positive and one negative, which are *contradictory results.*

We can conclude nothing from an equation of the form

$$x^3 - px + q = 0;$$

for, introducing the term $\pm 0 \cdot x^2$, it becomes

$$x^3 \pm 0 \cdot x^2 - px + q = 0,$$

which contains one permanence and two variations, whether we take the superior or inferior sign. Therefore, this equation may have its three roots real, viz., two positive and one negative; or, two of its roots may be imaginary and one negative, since its last term is positive (Art. 292).

Of the commensurable Roots of Numerical Equations.

295. Every equation in which the co-efficients are whole numbers, that of the first term being 1, will have whole numbers only for its commensurable roots.

For, let there be the equation

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0;$$

in which P, $Q \ldots T$, U, are whole numbers, and suppose that it were possible for one root to be an irreducible fraction $\frac{a}{b}$. Substituting this fraction for x, the equation becomes

$$\frac{a^m}{b^m} + P\frac{a^{m-1}}{b^{m-1}} + Q\frac{a^{m-2}}{b^{m-2}} + \ldots + T\frac{a}{b} + U = 0;$$

whence, multiplying both members by b^{m-1}, and transposing,

$$\frac{a^m}{b} = -Pa^{m-1} - Qa^{m-2}b - \ldots - Tab^{m-2} - Ub^{m-1}.$$

But the second member of this equation is composed of the sum of entire numbers, while the first is essentially fractional, for a and b being prime with respect to each other, a^m and b will also be prime with respect to each other (Art. 95), and hence this equality cannot exist; for, an irreducible fraction cannot be equal to a whole number. Therefore, it is impossible for any irreducible fraction to satisfy the equation.

Now, it has been shown (Art. 262), that an equation containing rational, but fractional co-efficients, can be transformed into another in which the co-efficients are whole numbers, that of the first term being 1. Hence, *the search for commensurable roots*, either entire or fractional, *can always be reduced to that for entire roots.*

296. This being the case, take the general equation

$$x^m + Px^{m-1} + Qx^{m-2} \ldots + Rx^3 + Sx^2 + Tx + U = 0,$$

and let a denote any entire number, positive or negative, which will satisfy it.

Since a is a root, we shall have the equation

$$a^m + Pa^{m-1} + \ldots + Ra^3 + Sa^2 + Ta + U = 0 \quad . \quad (1).$$

Now replace a by all the entire numbers, positive and negative, between 1 and the limit $+L$, and between -1 and $-L''$: those which verify the above equality will be roots of the equation. But these trials being long and troublesome, we will deduce from equation (1), other conditions equivalent to this, and more easily applied.

Transposing in equation (1) all the terms except the last, and dividing by a, we have,

$$\frac{U}{a} = -a^{m-1} - Pa^{m-2} - \ldots - Ra^2 - Sa - T \quad . \quad . \quad . \quad (2).$$

Now, the second member of this equation is an entire number; hence, $\frac{U}{a}$ must be an entire number; therefore, *the entire roots of the equation are comprised among the divisors of the last term.*

Transposing $-T$ in equation (2), dividing by a, and making

$$\frac{U}{a} + T = T', \quad \text{we have,}$$

$$\frac{T'}{a} = -a^{m-2} - Pa^{m-3} \ldots - Ra - S \quad . \quad . \quad . \quad . \quad (3).$$

The second member of this equation being entire, $\frac{T'}{a}$, that is, *the quotient of*

$$\frac{U}{a} + T \quad \text{by} \quad a,$$

is an entire number.

Transposing the term $-S$ and dividing by a, we have, by supposing

$$\frac{T'}{a} + S = S',$$

$$\frac{S'}{a} = -a^{m-3} - Pa^{m-4} - \ldots - R \ldots (4).$$

The second member of this equation being entire, $\frac{S'}{a}$, that is, *the quotient of*

$$\frac{T'}{a} + S \quad \text{by} \quad a,$$

is an entire number.

By continuing to transpose the terms of the second member into the first, we shall, after $m-1$ transformations, obtain an equation of the form,

$$\frac{Q'}{a} = -a - P.$$

Then, transposing the term $-P$, dividing by a, and making $\frac{Q'}{a} + P = P'$, we have $\frac{P'}{a} = -1$, or $\frac{P'}{a} + 1 = 0$.

This equation, which results from the continued transformations of equation (1), expresses the *last condition which it is requisite for* the entire number a to fulfil, in order that it may be known to be a root of the equation.

297. From the preceding conditions we conclude that, when an entire number a, positive or negative, is a root of the given equation, *the quotient of the last term, divided by* a, *is an entire number.*

Adding to this quotient the co-efficient of x^1, *the sum will be exactly divisible by a.*

Adding the co-efficient of x^2 to this last quotient, and again dividing by a, *the new quotient must also be entire;* and so on.

Finally, adding the co-efficient of the second term, that is, of x^{m-1}, to the preceding quotient, *the quotient of this sum divided by* a, *must be equal to* -1; hence, *the result of the addition of* 1, *which is the co-efficient of* x^m, *to the preceding quotient, must be equal to* 0.

Every number which will satisfy these conditions will be a root, and those which do not satisfy them should be rejected.

All the entire roots may be determined at the same time, by the following

RULE.

After having determined all the divisors of the last term, write those which are comprehended between the limits $+L$ *and* $-L''$ *upon the same horizontal line; then underneath these divisors write the quotients of the last term by each of them*

Add the co-efficient of x^1 *to each of these quotients, and write the sums underneath the quotients which correspond to them. Then divide these sums by each of the divisors, and write the quotients underneath the corresponding sums, taking care to reject the fractional quotients and the divisors which produce them;* and so on.

When there are terms wanting in the proposed equation, their co-efficients, which are to be regarded as equal to 0, must be taken into consideration.

EXAMPLES.

1. What are the entire roots of the equation,

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0 ?$$

A superior limit of the positive roots of this equation (Art. 284), is $13 + 1 = 14$. The co-efficient 48 need not be considered, since the last two terms can be put under the form $16(x - 3)$; hence, when $x > 3$, this part is essentially positive.

A superior limit of the negative roots (Art. 286), is

$$-(1 + \sqrt{48}), \quad \text{or} \quad -8.$$

Therefore, the divisors of the last term which may be roots, are 1, 2, 3, 4, 6, 8, 12; moreover, neither $+1$, nor -1, will satisfy the equation, because the co-efficient -48 is itself greater than the sum of all the others: we should therefore try only the *positive divisors* from 2 to 12, and the *negative divisors* from -2 to -6 inclusively.

By observing the rule given above, we have

12,	8,	6,	4,	3,	2,	− 2,	− 3,	− 4,	− 6
− 4,	− 6,	− 8,	− 12,	− 16,	− 24,	+ 24,	+ 16,	+ 12,	+ 8
+ 12,	+ 10,	+ 8,	+ 4,	0,	− 8,	+ 40,	+ 32,	+ 28,	+ 24
+ 1,	..,	..,	+ 1,	0,	− 4,	− 20,	..,	− 7,	− 4
− 12,	..,	..,	− 12,	− 3,	− 17,	− 33,	..,	− 20,	− 17
− 1,	..,	..,	− 3,	..,	..,	..,	..,	+ 5,	
− 2,	..,	..,	− 4,	..,	..,	..,	..,	+ 4,	..
..,	..,	..,	− 1,	..,	..,	..,	..,	− 1,	..

The *first* line contains the divisors, the *second* contains the quotients arising from the division of the last term -48, by each of the divisors. The *third* line contains these quotients, each augmented by the co-efficient $+16$; and the *fourth*, the quotients of these sums by each of the divisors; this second condition excludes the divisors $+8$, $+6$, and -3.

The *fifth* contains the preceding line of quotients, each augmented by the co-efficient -13, and the *sixth* contains the quotients of these sums by each of the divisors; the third condition excludes the divisors 3, 2, -2, and -6.

Finally, the *seventh* is the third line of quotients, each augmented by the co-efficient -1, and the *eighth* contains the quotients of these sums by each of the divisors. The divisors $+4$ and -4 are the only ones which give -1; hence, $+4$ and -4 are the only entire roots of the equation.

In fact, if we divide

$$x^4 - x^3 - 13x^2 + 16x - 48,$$

by the product $(x-4)(x+4)$, or $x^2 - 16$, the quotient will be $x^2 - x + 3$, which placed equal to zero, gives

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-11};$$

therefore, the four roots are

$$4, \quad -4, \quad \frac{1}{2} + \frac{1}{2}\sqrt{-11} \quad \text{and} \quad \frac{1}{2} - \frac{1}{2}\sqrt{-11}.$$

2. What are the entire roots of the equation

$$x^4 - 5x^3 + 25x - 21 = 0\,?$$

3. What are the entire roots of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0\,?$$

4. What are the entire roots of the equation

$$9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0\,?$$

Sturm's Theorem.

298. The object of this theorem is to explain a method of determining the number and places of the real roots of equations involving but one unknown quantity.

Let $$X = 0 \quad - \; - \; - \; - \quad (1),$$

represent an equation containing the single unknown quantity x; X being a polynomial of the m^{th} degree with respect to x, the co-efficients of which are all real. If this equation should have equal roots, they may be found and divided out as in Art. 269, and the reasoning be applied to the equation which would result. We will therefore suppose $X = 0$ to have no equal roots.

299. Let us denote the first derived polynomial of X by X_1, and then apply to X and X_1 a process similar to that for finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders as at first obtained, *we change their signs, and take care also, in preparing for the division, neither to introduce nor reject any factor except a positive one.*

If we denote the several remainders, in order, after their signs have been changed, by X_2, $X_3 \ldots X_r$, which are read X second, X third, &c., and denote the corresponding quotients by Q_1, Q_2 .. Q_{r-1}, we may then form the equations

$$X = X_1 Q_1 - X_2 \quad - \; - \; - \; - \quad (2).$$

$$\left.\begin{array}{l} X_1 = X_2 Q_2 - X_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{n-1} = X_n Q_n - X_{n+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{r-2} = X_{r-1} Q_{r-1} - X_r \end{array}\right\} \quad - \; - \; - \quad (3).$$

Since by hypothesis, $X = 0$ has no equal roots, no common divisor can exist between X and X_1 (Art. 267). The last remainder $-X_r$, will therefore *be different from zero*, and *independent* of x.

300. Now, let us suppose that a number p has been substituted for x in each of the expressions X, X_1, $X_2 \ldots X_{r-1}$; and that the signs of the results, together with the sign of X_r, are arranged in a line one after the other: also that another number q, greater than p, has been substituted for x, and the signs of the results arranged in like manner.

Then will the number of variations in the signs of the first arrangement, diminished by the number of variations in those of the second, denote the exact number of real roots comprised between p *and* q.

301. The demonstration of this truth mainly depends upon the three following properties of the expressions X, $X_1 \ldots X_n$, &c.

I. *If any number be substituted for* x *in these expressions, it is impossible that any two consecutive ones can become zero at the same time.*

For, let X_{n-1}, X_n, X_{n+1}, be any three consecutive expressions. Then among equations (3), we shall find

$$X_{n-1} = X_n Q_n - X_{n+1} \ldots . (4),$$

from which it appears that, if X_{n-1} and X_n should both become 0 for a value of x, X_{n+1} would be 0 for the same value; and since the equation which follows (4) must be

$$X_n = X_{n+1} Q_{n+1} - X_{n+2},$$

we shall have $X_{n+2} = 0$ for the same value, and so on until we should find $X_r = 0$, which cannot be; hence, X_{n-1} and X_n cannot both become 0 for the same value of x.

II. By an examination of equation (4), we see that if X_n becomes 0 for a value of x, X_{n-1} and X_{n+1} must have contrary signs, that is,

If any one of the expressions is reduced to 0 by the substitution of a value for x, *the preceding and following ones will have contrary signs for the same value.*

III. Let us substitute $a + u$ for x in the expressions X and X_1, and designate by U and U_1 what they respectively become under this supposition. Then (Art. 264), we have

$$\left.\begin{aligned} U &= A + A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A_1 + A'_1 u + A''_1\frac{u^2}{2} + \&c. \end{aligned}\right\} \quad \cdot\cdot\cdot \quad (5),$$

in which A, A', A'', &c., are the results obtained by the substitution of a for x, in X and its derived polynomials; and A_1, A'_1, &c., are similar results derived from X_1. If, now, a be a root of the proposed equation $X = 0$, then $A = 0$, and since A' and A_1 are each derived from X_1, by the substitution of a for x, we have $A' = A_1$, and equations (5) become

$$\left.\begin{aligned} U &= A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A' + A'_1 u + \&c. \end{aligned}\right\} \quad \cdot\cdot\cdot \quad (6).$$

Now, the arbitrary quantity u may be taken so small that the signs of the values of U and U_1 will depend upon the signs of their first terms (Art. 276); that is, they will be alike when u is positive, or when $a + u$ is substituted for x, and unlike when u is negative or when $a - u$ is substituted for x. Hence,

If a number insensibly less than one of the real roots of $\mathrm{X} = 0$ *be substituted for* x *in* X *and* X_1, *the results will have contrary signs; and if a number insensibly greater than this root be substituted, the results will have the same sign.*

302. Now, let any number as k, algebraically less, that is, nearer equal to $-\infty$, than any of the real roots of the several equations

$$X = 0, \quad X_1 = 0 \; . \; . \; . \; X_{r-1} = 0,$$

be substituted for x in the expressions X, X_1, X_2, &c., and the signs of the several results arranged in order; then, let x be increased by insensible degrees, until it becomes equal to h, the least of all the roots of the equations. As there is no

root of either of the equations between k and h, none of the signs can change while x is less than h (Art. 277), and the number of variations and permanences in the several sets of results, will remain the same as in those obtained by the first substitution.

When x becomes equal to h, *one* or *more* of the expressions X, X, &c., will reduce to 0. Suppose X_n becomes 0. Then, as by the first and second properties above explained, neither X_{n-1} nor X_{n+1} can become 0 at the same time, but *must have* contrary signs, it follows that in passing from one to the other (omitting $X_n = 0$), there will be *one* and *only one* variation; and since their signs have not changed, one must be *the same as*, and the other *contrary to*, that of X_n, both before and after it becomes 0; hence, in passing over the three, either just before X_n becomes 0 or just after, there is *one* and *only one* variation. Therefore, the reduction of X_n to 0 neither increases nor diminishes the number of variations; and this will evidently be the case, although several of the expressions X_1, X_2, &c., should become 0 at the same time.

If $x = h$ should reduce X to 0, then h is the least real root of the proposed equation, which root we denote by a; and since by the third property, just before x becomes equal to a, the signs of X and X_1 are contrary, *giving a variation*, and just after passing it (before x becomes equal to a root of $X_1 = 0$), the signs are the same, *giving a permanence instead*, it follows that in passing this root *a variation is lost.*

In the same way, increasing x by insensible degrees from $x = a + u$ until we reach the root of $X = 0$ next in order, it is plain that no variation will be lost or gained in passing any of the roots of the other equations, but that in passing this root, for the same reason as before, another variation will be lost, and so on for each real root between k and the number last substituted, as g, a variation will be lost until x has been increased beyond the greatest real root, *when no more can be lost or gained.* Hence, the *excess* of the number of variations

obtained by the substitution of k over those obtained by the substitution of g, will be equal to the number of real roots comprised between k and g.

It is evident that the same course of reasoning will apply when we commence with any number p, whether less than all the roots or not, and gradually increase x until it equals any other number q. The fact enunciated in Art. 299 is therefore established.

303. In seeking the number of roots comprised *between* p and q, should either p or q reduce any of the expressions X_1, X_2, &c., to 0, the result will not be affected by their omission, since the number of variations will be the same.

Should p reduce X to 0, then p is a root, but not one of those sought; and as the substitution of $p + u$ will give X and X_1 the same sign, the number of variations to be counted will not be affected by the omission of $X = 0$.

Should q reduce X to 0, then q is also a root, but not one of those sought; and as the substitution of $q - u$ will give X and X_1 contrary signs, one variation must be counted in passing from X to X_1.

304. If in the application of the preceding principles, we observe that any one of the expressions X_1, X_2 . . . &c., X_n for instance, will preserve the same sign for all values of x in passing from p to q, inclusively, it will be unnecessary to use the succeeding expressions, or even to deduce them. For, as X_n preserves the same sign during the successive substitutions, it is plain that the same number of variations will be lost among the expressions X, X_1, &c. . . . ending with X_n as among all including X_r. Whenever then, in the course of the division, it is found that by placing any of the remainders equal to 0, an equation is obtained with imaginary roots only (Art. 291), it will be useless to obtain any of the succeeding remainders. This principle will be found very useful in the solution of numerical examples.

305. As all the real roots of the proposed equation are necessarily included between $-\infty$ and $+\infty$, we may, by ascertaining the number of variations lost by the substitution of these, in succession, in the expressions X, X_1 . . . X_n, . . &c., readily determine the total number of such roots. It should be observed, that it will be only necessary to make these substitutions in the first terms of each of the expressions, as in this case the sign of the term will determine that of the entire expression (Art. 282).

Having found the number of real roots, if we subtract this number from the highest exponent of the unknown quantity, the remainder will be the number of imaginary roots (Art. 248).

306. Having thus obtained the total number of real roots, we may ascertain their places by substituting for x, in succession, the values 0, 1, 2, 3, &c., until we find an entire number which gives the same number of variations as $+\infty$. This will be the smallest superior limit of the positive roots in entire numbers.

Then substitute -1, -2, &c., until a negative number is obtained which gives the same number of variations as $-\infty$. This will be, numerically, the least superior limit of the negative roots in entire numbers. Now, by commencing with this limit and observing the number of variations lost in passing from each number to the next in order, we shall discover how many roots are included between each two of the consecutive numbers used, and thus, of course, know the entire part of each root. The decimal part may then be sought by some of the known methods of approximation.

EXAMPLES.

1. Let $$8x^3 - 6x - 1 = 0 = X.$$

The first derived polynomial (Art. 264), is

$$24x^2 - 6.$$

and since we may omit the positive factor 6, without affecting the sign, we may write

$$4x^2 - 1 = X_1.$$

Dividing X by X_1, we obtain for the first remainder, $-4x - 1$. Changing its sign, we have

$$4x + 1 = X_2.$$

Multiplying X_1 by the positive number 4, and then dividing by X_2, we obtain the second remainder -3; and by changing its sign

$$+3 = X_3.$$

The expressions to be be used are then

$$X = 8x^3 - 6x - 1, \quad X_1 = 4x^2 - 1, \quad X_2 = 4x + 1, \quad X_3 = +3.$$

Substituting $-\infty$ and then $+\infty$, we obtain the two following arrangements of signs:

$- + - + \ldots\ldots$ 3 *variations*,

$+ + + + \ldots\ldots$ 0 "

there are then *three* real roots.

If, now, in the same expressions we substitute 0 and $+1$, and then 0 and -1, for x, we shall obtain the three following arrangements:

For	$x = +1$	$+ + + +$	0 *variations.*
"	$x = \;\;0$	$- - + +$	1 "
"	$x = -1$	$- + - +$	3 "

As $x = +1$ gives the same number of variations as $+\infty$, and $x = -1$ gives the same as $-\infty$, $+1$ and -1 are the smallest limits in entire numbers. In passing from -1 to 0, *two* variations are lost, and in passing from 0 to $+1$, *one* variation is lost; hence, there are two negative roots between -1 and 0, and one positive root between 0 and $+1$.

2. Let $\qquad 2x^4 - 13x^2 + 10x - 19 = 0.$

If we deduce X, X_1, and X_2, we have the three expressions

$$X = 2x^4 - 13x^2 + 10x - 19,$$

$$X_1 = 4x^3 - 13x + 5,$$

$$X_2 = 13x^2 - 15x + 38.$$

If we place $X_2 = 0$, we shall find that both of the roots of the resulting equation are imaginary; hence, X_2 will be positive for all values of x (Art. 290). It is then useless to seek for X_3 and X_4.

By the substitution of $-\infty$ and $+\infty$ in X, X_1, and X_2, we obtain for the first, *two* variations, and for the second, *none;* hence, there are two real and two imaginary roots in the proposed equation.

3. Let $x^3 - 5x^2 + 8x - 1 = 0.$

4. $x^4 - x^3 - 3x^2 + x^2 - x - 3 = 0.$

5. $x^5 - 2x^3 + 1 = 0.$

Discuss each of the above equations.

307. In the preceding discussions we have supposed the equations to be given, and from the relations existing between the co-efficients of the different powers of the unknown quantity, have determined the number and places of the real roots; and, consequently, the number of imaginary roots.

In the equation of the second degree, we pointed out the relations which exist between the co-efficients of the different powers of the unknown quantity when the roots are real, and when they are imaginary (Art. 116).

Let us see if we can indicate corresponding relations among the co efficients of an equation of the third degree.

Let us take the equation,

$$x^3 + Px^2 + Qx + U = 0,$$

and by causing the second term to disappear (Art. 263), it will take the form,

$$x^3 + px + q = 0.$$

Hence, we have

$$X = x^3 + px + q,$$
$$X_1 = 3x^2 + p,$$
$$X_2 = -2px - 3q,$$
$$X_3 = -4p^3 - 27q^2.$$

In order that all the roots be real, the substitution of ∞ for x in the above expressions must give three permanences; and the substitution of $-\infty$ for x must give three variations. But the first supposition can only give three permanences when

$$-4p^3 - 27q^2 > 0;$$

that is, a positive quantity, a condition which requires that p be negative.

If, then, p be negative, we have, for $x = \infty$,

$$-4p^3 - 27q^2 > 0; \quad \text{that is, positive:}$$

or, $$4p^3 + 27q^2 < 0; \quad \text{that is, negative:}$$

hence, $$\frac{q^2}{4} + \frac{p^3}{27} < 0,$$ which requires that p be negative, and that $\frac{p^3}{27} > \frac{q^2}{4}$; conditions which indicate that the roots are all real.

Cardan's Rule for Solving Cubic Equations.

308. First, free the equation of its second term, and we have the form,

$$x^3 + px + q = 0 \quad \cdots\cdots \quad (1).$$

Take $$x = y + z;$$

then $$x^3 = y^3 + z^3 + 3yz(y + z);$$

or, by transposing, and substituting x for $y + z$, we have

$$x^3 - 3yz \, . \, x - (y^3 + z^3) = 0 \quad \cdots \quad (2);$$

and by comparing this with equation (1), we have

$$-3yz = p; \quad \text{and} \quad y^3 + z^3 = -q.$$

From the 1st, we have

$$z^3 = -\cdot\frac{p^3}{27y^3},$$

which, being substituted in the second, gives

$$y^3 - \frac{p^3}{27y^3} = -q;$$

or clearing of fractions, and reducing

$$y^6 + qy^3 = \frac{p^3}{27}.$$

Solving this trinomial equation (Art. 124), we have

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

and the corresponding value of z is

$$z = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

But since $x = y + z$, we have

$$x = \sqrt[3]{\left[-\frac{q}{2} + \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right]} + \sqrt[3]{\left[-\frac{q}{2} - \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right]}.$$

This is called *Cardan's formula.*

By examining the above formula, it will be seen, that it is inapplicable to the case, when the quantity

$$\frac{q^2}{4} + \frac{p^3}{27},$$

under the radical of the second degree, is negative; and hence is applicable only to the case where two of the roots are imaginary (Art. 307).

Having found the real root, divide both members of the given equation by the unknown quantity, minus this root (Art. 247); the result will be an equation of the second degree, the roots of which may be readily found.

EXAMPLES.

1. What are the roots of the equation

$$x^3 - 6x^2 + 10x = 8?$$

Ans. 4, $1+\sqrt{-1}$ $1-\sqrt{-1}$.

2. What are the roots of the equation

$$x^3 - 9x^2 + 28x = 30?$$

Ans. 3, $3+\sqrt{-1}$, $3-\sqrt{-1}$.

3. What are the roots of the equation

$$x^3 - 7x^2 + 14x = 20?$$

Ans. 5, $1+\sqrt{-3}$, $1-\sqrt{-3}$.

Preliminaries to Horner's Method.

309. Before applying the method of Horner to the solution of numerical equations, it will be necessary to explain,

1st. A modification of the method of multiplication, called the method by *Detached Co-efficients:*

2d. A modification of the method of division, called, also, the method by *Detached Co-efficients:*

3d. A second modification of the method of division, called *Synthetical Division:* and,

4th. The application of these methods of Division in the *Transformation of Equations.*

Multiplication by Detached Co-efficients.

310. When the multiplicand and multiplier are both homogeneous (Art. 26), and contain but two letters, if each be arranged according to the same letter, the literal part, in the several terms of the product, may be written immediately, since the exponent of the leading letter will go on decreasing from left to right by a constant number, and the sum of the exponents of both letters will be the same, in each of the terms.

EXAMPLES.

1. Let it be required to multiply

$$x^3 + x^2y + xy^2 + y^3 \quad \text{by} \quad x - y.$$

Since $x^3 \times x = x^4$, the terms of the product will be of the 4th degree, and since the exponents of x decrease by 1, and those of y increase by 1, we may write the literal parts thus,

$$x^4, \quad x^3y, \quad x^2y^2, \quad xy^3, \quad y^4.$$

In regard to the co-efficients, we have,

$$\begin{array}{lr} \text{Co-efficients of multiplicand, - - -} & 1+1+1+1. \\ \text{" \quad " multiplier, \quad -} & 1-1 \\ \hline & 1+1+1+1 \quad\; \\ & -1-1-1-1 \\ \hline \text{co-efficients of the product, \quad - -} & 1+0+0+0-1\,; \end{array}$$

and writing these co-efficients before the literal parts to which they belong, we have

$$x^4 + 0\,.\,x^3y + 0\,.\,x^2y^2 + 0\,.\,xy^3 - y^4 = x^4 - y^4.$$

2. Multiply $\quad 2a^3 - 3ab^2 + 5b^3 \quad$ by $\quad 2a^2 - 5b^2$.

In this example, the term a^2b in the multiplicand, and ab in the multiplier, are both wanting; that is, their co-efficients are 0. Supplying these co-efficients, and we have,

$$\begin{array}{lr} \text{Co-efficients of multiplicand, -} & 2+0-\;\;3+\;\;5 \qquad\qquad\quad \\ \text{" \quad " multiplier, - -} & 2+0-\;\;5 \qquad\qquad\qquad\qquad \\ \hline & 4+0-\;\;6+10 \qquad\qquad\quad \\ & -10-\;\;0+15-25 \\ \hline \text{co-efficients of the product, - -} & 4+0-16+10+15-25. \end{array}$$

Hence, the product is, $\quad 4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5$.

3. Multiply $\quad x^3 - 3x^2 + 3x - 1 \quad$ by $\quad x^2 - 2x + 1$.

4. Multiply $\quad y^2 - ya + \frac{1}{4}a^2 \quad$ by $\quad y^2 + ya - \frac{1}{4}a^2$.

REMARK.—The method by detached co-efficients is also applicable to the case, in which the multiplicand and multiplier contain but a single letter. The terms whose co-efficients are zero must be supplied, when wanting, as in the previous examples.

EXAMPLES.

1. What is the product of $a^4 + 3a^2 + 1$ by $a^2 - 3$?

2. What is the product of $b^2 - 1$ by $b + 2$?

Division by Detached Co-efficients.

311. When the dividend and divisor are both homogeneous and contain but two letters, the division may be performed by means of detached co-efficients, in the following manner:

1. Arrange the terms of the dividend and divisor according to a common letter.

2. Subtract the highest exponent of the leading letter of the divisor from the exponent of the leading letter of the dividend, and the remainder will be the exponent of the leading letter of the quotient.

3. The exponents of the letters in the other terms follow the same law of increase or decrease as the exponents in the corresponding terms of the dividend.

4. Write down for division the co-efficients of the different terms of the dividend and divisor, with their respective signs, supplying the deficiency of the absent terms with zeros.

5. Then divide the co-efficients of the dividend by those of the divisor, after the manner of algebraic division, and prefix the several quotients to their corresponding literal parts.

EXAMPLES.

2. Divide $8a^5 - 4a^4x - 2a^3x^2 + a^2x^3$ by $4a^2 - x^2$.

The literal part will be

$$a^3, \quad a^2x, \quad ax^2, \quad x^3;$$

and for the numerical co-efficients,

$$\begin{array}{rrrr|l}
8 & -4 & -2 & +1 & \underline{4+0-1;} \\
\underline{8} & \underline{+0} & \underline{-2} & & 2-1 \\
 & -4 & & +1 & \\
 & \underline{-4} & & \underline{+1} & \\
 & 0 & & 0 &
\end{array}$$

hence, the true quotient is $2a^3 - a^2x$; the co-efficients after -1, being each equal to zero.

3. Divide $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$ by $x^2 - 2ax - 2a^2$.

4. Divide $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$ by $2a^2 - 3ax + 4x^2$.

REMARK.—The method by detached co-efficients is also applicable to all cases in which the dividend and divisor contain but a single letter. The terms whose co-efficients are zero, must be supplied, when wanting, as in the previous examples.

EXAMPLES.

1. Let it be required to divide

$$6a^4 - 96 \quad \text{by} \quad 3a - 6.$$

The dividend, in this example, may be written under the form,

$$6a^4 + 0 \,.\, a^3 + 0 \,.\, a^2 + 0 \,.\, a - 96a^0.$$

Dividing a^4 by a, we have a^3 for the literal part of the first term of the quotient; hence, the form of the quotient is

$$a^3, \quad a^2, \quad a, \quad a^0.$$

For the co-efficients, we have,

$$\begin{array}{l||l} 6 + 0 + 0 + 0 - 96 & 3 - 6 \\ \hline \underline{6 - 12} & 2 + 4 + 8 + 16 \quad \text{quotient;} \end{array}$$

hence, the true quotient is,

$$2a^3 + 4a^2 + 8a + 16.$$

Synthetical Division.

312. In the common method of division, each term of the divisor is multiplied by the first term of the quotient, and the products subtracted from the dividend; but the subtractions are performed by first changing the sign of each product, and then adding. If, therefore, the signs of the divisor were first changed, we should obtain the same result by adding the products, instead of subtracting as before, and the same for any subsequent operation.

By this process, the second dividend would be the same as by the common method. But since the second term of the quotient is found by dividing the first term of the second dividend by the first term of the divisor; and since the sign of the latter has been changed, it follows, that the sign of the second term of the quotient will also be changed.

To avoid this change of sign, the sign of the first term of the divisor is left unchanged, and the products of all the terms of the quotient by the first term of the divisor, are omitted; be cause, in the usual method, the first terms in each successive dividend are cancelled by these products.

Having made the first term of the divisor 1 before commenc ing the operation, and omitting these several products, the co-efficient of the first term of any dividend will be the co-efficient of the succeeding term of the quotient. Hence, the co-efficients in the quotient are, respectively, the co-efficients of the first terms of the successive dividends.

The operation, thus simplified, may be farther abridged by omitting the successive additions, except so much only as may be necessary to show the first term of each dividend; and also, by writing the products of the several terms of the quotient by the modified divisor, diagonally, instead of horizontally, the first product falling under the second term of the dividend.

Hence, the following

RULE.

I. *Divide the divisor and dividend by the co-efficient of the first term of the divisor, when that co-efficient is not* 1.

II. *Write, in a horizontal line, the co-efficients of the dividend, with their proper signs, and place the co-efficients of the divisor, with all their signs changed, except the first, on the right.*

III. *Divide as in the method by detached co-efficients, except that no term of the quotient is multiplied by the first term of the divisor, and that all the products are written diagonally to the right, under the terms of the dividend to which they correspond.*

IV. *The first term of the quotient is the same as that of the dividend; the second term is the sum of the numbers in the second column; the third term, the sum of the numbers in third column, and so on, to the right.*

V. *When the division can be exactly made, columns will be found at the right, whose sums will be zero: when the division is not exact, continue the operation until a sufficient degree of approximation is attained. Having found the co-efficients, annex to them the literal parts.*

EXAMPLES.

1. Divide

$$a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5 \quad \text{by} \quad a^2 - 2ax + x^2.$$

$$\begin{array}{rrrrrr|l}
1 & -5 & +10 & -10 & +5 & -1 & \;1 + 2 - 1 \\
 & 2 & -6 & +6 & -2 & & \;\overline{1 - 3 + 3 - 1} \\
 & & -1 & +3 & -3 & +1 & \\
\hline
1 & -3 & +3 & -1 & 0 & 0. &
\end{array}$$

Hence, the quotient is

$$a^3 - 3a^2x + 3ax^2 - x^3.$$

REMARK.—The first term of the divisor being always 1, need not be written. The first term of the quotient is the same as that of the dividend.

2. Divide

$$x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 13x + 5 \quad \text{by} \quad x^4 - 2x^3 + 4x^2 - 2x + 1.$$

$$\begin{array}{rrrrrrr|l}
1 & -5 & +15 & -24 & +27 & -13 & +5 & \;1 + 2 - 4 + 2 - 1 \\
 & +2 & -6 & +10 & & & & \;\overline{1 - 3 + 5} \\
 & & -4 & +12 & -20 & & & \\
 & & & +2 & -6 & +10 & & \\
 & & & & -1 & +3 & -5 & \\
\hline
1 & -3 & +5 & 0 & 0 & 0 & 0. &
\end{array}$$

Hence, the quotient is $x^2 - 3x + 5$.

3. Divide

$$a^5 + 2a^4b + 3a^3b^2 - a^2b^3 - 2ab^4 - 3b^5 \quad \text{by} \quad a^2 + 2ab + 3b^2.$$

Ans. $a^3 + 0 . a^2b + 0 . ab^2 - b^3 = a^3 - b^3$.

4. Divide $1 - x$ by $1 + x$. *Ans.* $1 - 2x + 2x^2 - 2x^3 + \&c.$

5. Divide 1 by $1 - x$. *Ans.* $1 + x + x^2 + x^3 + \&c.$

6. Divide $x^7 - y^7$ by $x - y$.
Ans. $x^6 + x^5 y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$.

7. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$ by $a^3 - 3a^2x + 3ax^2 - x^3$.
Ans. $a^3 + 3a^2x + 3ax^2 + x^3$.

313. *To transform an equation into another whose roots shall be the roots of the proposed equation, increased or diminished by a given quantity.*

A method of solving this problem has already been explained (Art. 264); but the process is tedious. We shall now explain a more simple method of finding the transformed equation.

Let it be required to transform the equation

$$ax^m + Px^{m-1} + Qx^{m-2} \;.\;.\;.\;.\; Tx + U = 0$$

into another whose roots shall be less than the roots of this equation by r.

If we write $y + r$ for x, and develop, and arrange the terms with reference to y, we shall have

$$ay^m + P'y^{m-1} + Q'y^{m-2} \;.\;.\;.\;.\; + T'y + U' = 0 \quad - \; - \; - \quad (1).$$

But since $y = x - r$, equation (1), may take the form

$$a(x-r)^m + P'(x-r)^{m-1} + Q'(x-r)^{m-2} \;.\;.\;.\;.\; T'(x-r) + U' = 0 \quad (2),$$

which, when developed, must be identical with the given equation. For, since $y + r$ was substituted for x in the proposed equation, and then $x - r$ for y in the transformed equation, we must necessarily have returned to the given equation. Hence, we have

$$a(x-r)^m + P'(x-r)^{m-1} + Q'(x-r)^{m-2} \;.\;.\;.\; T'(x-r) + U'$$
$$= ax^m + Px^{m-1} + Qx^{m-2} \;.\;.\;.\; Tx + U = 0.$$

If now we divide the first member by $x - r$, the quotient will be

$$a(x-r)^{m-1} + P'(x-r)^{m-2} + Q'(x-r)^{m-3} \;.\;.\;.\; T',$$

and the remainder U'.

But since the second member is identical with the first, the very same quotient and the same remainder would arise, if the second member were divided by $x - r$: hence,

If the first member of the given equation be divided by the unknown quantity minus the number which expresses the difference between the roots, the remainder will be the absolute term of the transformed equation.

Again, if we divide the quotient thus obtained: viz.,

$$a(x-r)^{m-1} + P'(x-r)^{m-2} + Q'(x-r)^{m-3} \ldots T'$$

by $x - r$, the remainder will be T', the co-efficient of the term last but one of the transformed equation; and a similar result would be obtained by again dividing the resulting quotient by $x - r$. Hence, by successive divisions of the polynomial in the first member of the given equation and the quotients which result, by $x - r$, we shall obtain all the co-efficients of the transformed equation, in an inverse order.

REMARK.—When there is an absent term in the given equation, its place must be supplied by a 0.

EXAMPLES.

Transform the equation

$$5x^4 - 12x^3 + 3x^2 + 4x - 5 = 0$$

into another whose roots shall each be less than those of the given equation by 2.

First Operation.

$$\begin{array}{rrrrr|l}
5x^4 & -12x^3 & +3x^2 & +4x & -5 & \;x-2 \\
\cline{6-6}
5x^4 & -10x^3 & & & & \;5x^3 - 2x^2 - x + 2 \\
\cline{1-2}
 & -2x^3 & +3x^2 & & & \\
 & -2x^3 & +4x^2 & & & \\
\cline{2-3}
 & & -x^2 & +4x & & \\
 & & -x^2 & +2x & & \\
\cline{3-4}
 & & & 2x & -5 & \\
 & & & 2x & -4 & \\
\cline{4-5}
 & & & & -1 & \text{1st remainder}
\end{array}$$

Second Operation.

$$\begin{array}{l|l} 5x^3 - 2x^2 - x + 2 & x - 2 \\ \cline{2-2} \underline{5x^3 - 10x^2} & 5x^2 + 8x + 15 \\ \qquad\quad 8x^2 - x & \\ \qquad\quad \underline{8x^2 - 16x} & \\ \qquad\qquad\quad 15x + 2 & \\ \qquad\qquad\quad \underline{15x - 30} & \\ \qquad\qquad\qquad\quad 32 \quad \text{2d remainder.} & \end{array}$$

Third Operation.

$$\begin{array}{l|l} 5x^2 + 8x + 15 & x - 2 \\ \cline{2-2} \underline{5x^2 - 10x} & 5x + 18 \\ \qquad\quad 18x + 15 & \\ \qquad\quad \underline{18x - 36} & \\ \qquad\qquad\quad 51 \quad \text{3d remainder.} & \end{array}$$

Fourth Operation.

$$\begin{array}{l|l} 5x + 18 & x - 2 \\ \cline{2-2} \underline{5x - 10} & 5 \\ \qquad\; 28 \quad \text{4th remainder.} & \end{array}$$

Therefore, the transformed equation is

$$5y^4 + 28y^3 + 51y^2 + 32y - 1 = 0.$$

This laborious operation can be avoided by the synthetical method of division (Art. 312).

Taking the same example, and recollecting that in the synthetical method, the first term of the divisor not being used, may be omitted, and that the first term of the quotient, by which the modified divisor is to be multiplied for the first term of the product, is always the first term of the dividend; the whole of the work may be thus arranged:

$$\begin{array}{rrrrr|l} 5 & -12 & +3 & +4 & -5 & 2 \\ & 10 & -4 & -2 & 4 & \\ \hline & -2 & -1 & 2 & -1 & \therefore U' = -1 \\ & 10 & 16 & 30 & & \\ \cline{2-4} & 8 & 15 & 32 & & \therefore T' = 32 \\ & 10 & 36 & & & \\ \cline{2-3} & 18 & 51 & & & \therefore Q' = 51 \\ & 10 & & & & \\ \cline{2-2} & 28 & & & & \therefore P' = 28; \end{array}$$

for it is plain that the first remainder will fall under the absolute term, the second under the term next to the left, and so on. Hence, the transformed equation is

$$5y^4 + 28y^3 + 51y^2 + 32y - 1 = 0.$$

2. Find the equation whose roots are less by 1.7 than those of the equation

$$x^3 - 2x^2 + 3x - 4 = 0.$$

First, find an equation whose roots are less by 1.

$$\begin{array}{rrrr|l}
1 & -2 & +3 & -4 & \| 1 \\
 & 1 & -1 & 2 & \\
\hline
 & -1 & 2 & -2 & \\
 & 1 & 0 & & \\
\hline
 & 0 & 2 & & \\
 & 1 & & & \\
\hline
 & 1 & & &
\end{array}$$

We have thus found the co-efficients of the terms of an equation whose roots are less by 1 than those of the given equation: the equation is

$$x^3 + x^2 + 2x - 2 = 0;$$

and now by finding a new equation whose roots are less than those of the last by .7, we shall have the required equation: thus,

$$\begin{array}{rrrr|l}
1 & +1 & +2 & -2 & \| .7 \\
 & .7 & 1.19 & 2.233 & \\
\hline
 & 1.7 & 3.19 & .233 & \\
 & .7 & 1.68 & & \\
\hline
 & 2.4 & 4.87 & & \\
 & .7 & & & \\
\hline
 & 3.1 & & &
\end{array}$$

hence, the required equation is

$$y^3 + 3.1y^2 + 4.87y + .233 = 0.$$

This latter operation can be continued from the former, without arranging the co-efficients anew. The operations have been explained separately, merely to indicate the several steps in the

transformation, and to point out the equations, at each step resulting from the successive diminution of the roots. Combining the two operations, we have the following arrangement:

1	-2	$+3$	$-4(1.7$; or,
	1	-1	2
	-1	2	-2
	1	0	2.233
	0	2	.233
	1	1.19	
	1.7	3.19	
	.7	1.68	
	2.4	4.87	
	.7		
	3.1		

1	-2	$+3$	$-4(1.7$
	1.7	$-.51$	4.233
	$-.3$	2.49	.233
	1.7	2.38	
	1.4	4.87	
	1.7		
	3.1		

We see, by comparison, that the above results are the same as those obtained by the preceding operations.

3. Find the equation whose roots shall be less by 1 than the roots of

$$x^3 - 7x + 7 = 0.$$

Ans. $y^3 + 3y^2 - 4y + 1 = 0.$

4. Find the equation whose roots shall be less by 3 than the roots of the equation

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0.$$

Ans. $y^4 + 9y^3 + 12y^2 - 14y = 0.$

5. Find the equation whose roots shall be less by 10 than the roots of the equation

$$x^4 + 2x^3 + 3x^2 + 4x - 12340 = 0.$$

Ans. $y^4 + 42y^3 + 663y^2 + 4664y = 0.$

6. Find the equation whose roots shall be less by 2 than the roots of the equation

$$x^5 + 2x^3 - 6x^2 - 10x = 0.$$

Ans. $y^5 + 10y^4 + 42y^3 + 86y^2 + 70y + 4 = 0.$

Horner's Method of approximating to the Real Roots of Numerical Equations.

314. The method of approximating to the roots of a numerical equation of any degree, discovered by the English mathematician W. G. Horner, Esq., of Bath, is a process of very remarkable simplicity and elegance.

The process consists, simply, in a succession of transformations of one equation to another, each transformed equation, as it arises, having its roots equal to the difference between the true value of the roots of the given equation, and the part of the root expressed by the figures already found. Such figures of the root are called the *initial figures.* Let

$$V = x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0 \quad - \quad - \quad - \quad (1)$$

be any equation, and let us suppose that we have found a part of one of the roots, which we will denote by m, and denote the remaining part of the root by r.

Let us now transform the given equation into another, whose roots shall be less by m, and we have (Art. 313),

$$V' = r^m + P'r^{m-1} + Q'r^{m-2} \dots + T'r + U' = 0 \quad (2).$$

Now, when r is a very small fraction, all the terms of the second member, except the last two, may be neglected, and the first figure, in the value of r, may be found from the equation

$$T'r + U' = 0; \text{ giving } -r = \frac{U'}{T'}; \text{ or } r = -\frac{U'}{T'}; \text{ hence,}$$

The first figure of r *is the first figure of the quotient obtained by dividing the absolute term of the transformed equation by the penultimate co-efficient.*

If, now, we transform equation (2) into another, whose roots shall be less than those of the previous equation by the first figure of r, and designate the remaining part by s, we shall have,

$$V'' = s^m + P''s^{m-1} + Q''s^{m-2} \dots + T''s + U'' = 0,$$

the roots of which will be less than those of the given equation by $m +$ the first figure of r. The first figure in the value of s is found from the equation,

$$T''s + U'' = 0, \quad \text{giving} \quad s = \frac{U''}{T''}.$$

We may thus continue the transformations at pleasure, and each one will evolve a new figure of the root. Hence, to find the roots of numerical equations.

I. *Find the number and places of the real roots by Sturms' theorem, and set the negative roots aside.*

II. *Transform the given equation into another whose roots shall be less than those of the given equation, by the initial figure or figures already found: then, by Sturms' theorem, find the places of the roots of this new equation, and the first figure of each will be the first decimal place in each of the required roots.*

III. *Transform the equation again so that the roots shall be less than those of the given equation, and divide the absolute term of the transformed equation by the penultimate co-efficient, which is called the trial divisor, and the first figure of the quotient will be the next figure of the root.*

IV. *Transform the last equation into another whose roots shall be less than those of the previous equation by the figure last found, and proceed in a similar manner until the root be found to the required degree of accuracy.*

REMARK I.—This method is one of approximation, and it may happen that the rejection of the terms preceding the penultimate term will affect the quotient figure of the root. To avoid this source of error, find the first decimal places of the root, also, by the theorem of Sturm, as in example 4, page 399, and when the results coincide for two consecutive places of decimals, those subsequently obtained by the divisors may be relied on.

REMARK II.—When the decimal portion of a negative root is to be found, first transform the given equation into another by changing the signs of the alternate terms (Art. 280), and then find the decimal part of the corresponding positive root of this new equation.

III. When several decimal places are found in the root, the operation may be shortened according to the method of contractions indicated in the examples.

314. Let us now work one example in full. Let us take the equation of the third degree,

$$x^3 - 7x + 7 = 0.$$

By Sturm's rule, we have the functions (Art. 299),

$$\begin{aligned} X &= x^3 - 7x + 7 \\ X_1 &= 3x^2 - 7 \\ X_2 &= 2x - 3 \\ X_3 &= +\ 1. \end{aligned}$$

Hence, for $x = \infty$, we have $+ \ + \ + \ +$ no variation,

$x = -\infty$ " $- \ + \ - \ +$ three variations;

therefore, the equation has *three* real roots, two positive and one negative.

To determine the initial figures of these roots, we have

for $x = 0$	$\ldots + - - +$	for $x = 0$	$\ldots + - - +$
$x = 1$	$\ldots + - - +$	$x = -1$	$\ldots + - - +$
$x = 2$	$\ldots + + + +$	$x = -2$	$\ldots + + - +$
		$x = -3$	$\ldots + + - +$
		$x = -4$	$\ldots - + - +$

hence there are two roots between 1 and 2, and one between -3 and -4.

In order to ascertain the first figures in the decimal parts of the two roots situated between 1 and 2, we shall transform the preceding functions into others, in which the value of x is diminished by 1. Thus, for the function X, we have this operation:

$$\begin{array}{l} 1 + 0 - 7 + 7\ (1 \\ \quad\ \ 1 + 1 - 6 \\ \hline \quad\ \ 1 - 6 + 1 \\ \quad\ \ 1 + 2 \\ \hline \quad\ \ 2 - 4 \\ \quad\ \ 1 \\ \hline \quad\ \ 3 \end{array}$$

And transforming the others in the same way, we obtain the functions

$$\begin{aligned} Y &= y^3 + 3y^2 - 4y + 1; \\ Y_1 &= 3y^2 + 6y - 4; \\ Y_2 &= 2y - 1; \\ Y_3 &= +\ 1. \end{aligned}$$

Let $y = .1$	we have	$+\ -\ -\ +$	two variations,
$y = .2$	"	$+\ -\ -\ +$	"
$y = .3$	"	$+\ -\ -\ +$	"
$y = .4$	"	$-\ -\ -\ +$	one variation,
$y = .5$	"	$-\ -\ \mp\ +$	"
$y = .6$	"	$-\ +\ +\ +$	"
$y = .7$	"	$+\ +\ +\ +$	no variation.

Therefore the initial figures of the two positive roots are 1.3, 1.6.

Let us now find the decimal part of the first root.

1 +0	−7	+7 (1.356895867
1	1	−6
1	−6	*1
1	2	−.903
2	*−4	**.097
1	.99	−.086625
*3.3	−3.01	***.010375
3	1.08	−.009048984
3.6	**−1.93	****.001326016
3	.1975	−.001184430
**3.95	−1.7325	.000141586
5	.2000	−.000132923
4.00	***−1.5325	.000008663
5	.024336	−.000007382
***4.056	−1.508164	.000001281
6	.024372	−.000001181
4.062	****−1.483792	.000000100
6	.003254	−.000000089
****4.0688	−1.480538	.000000011
8	.003254	−.000000010
4.0696	−1.47728	1
	.00036	
	−1.47692	
	00036	
	−1.44765	

The operations in the example are performed as follows:

1st. We find the places and the initial figures of the positive roots, to include the first decimal place by Sturms' theorem.

2d. Then to find the decimal part of the first positive root, we arrange the co-efficients, and perform a succession of transformations by Synthetical Division, which must begin with the initial figures already known.

We first transform the given equation into another whose roots shall be less by 1. The co-efficients of this new equation are, 1, 3, -4 and 1, and are all, except the first, marked by a star. The root of this transformed equation, corresponding to the root sought of the given equation, is a decimal fraction of which we know the first figure 3.

We next transform the last equation into another whose roots are less by three-tenths, and the co-efficients of the new equation are each marked by two stars.

The process here changes, and we find the next figure of the root by dividing the absolute term .097 by the penultimate co-efficient -1.93, giving .05 for the next figure of the root.

We again transform the equation into another whose roots shall be less by .05, and the co-efficients of the new equation are marked by three stars.

We then divide the absolute term, .010375 by the penultimate co-efficient, -1.5325, and obtain .006, the next figure of the root: and so on for other figures.

In regard to the contractions, we may observe that, having decided on the number of decimal places to which the figures in the root are to be carried, we need not take notice of figures which fall to the right of that number in any of the dividends. In the example under consideration, we propose to carry the operations to the 9th decimal place of the root; hence, we may reject all the decimal places of the dividends after the 9th.

The fourth dividend, marked by four stars, contains nine decimal places, and the next dividend is to contain no more.

But the corresponding quotient figure 8, is the fourth figure from the decimal point; hence, at this stage of the operation, all the places of the divisor, after the 5th, may be omitted, since the 5th, multiplied by the 4th, will give the 9th order of decimals. Again: since each new figure of the root is removed one place to the right, one additional figure, in each subsequent divisor, may be omitted. The contractions, therefore, begin by striking off the 2 in the 4th divisor.

In passing from the first column to the second, in the next operation, we multiply by .0008; but since the product is to be limited to five decimal places, we need take notice of but one decimal place in the first column; that is, in the first operation of contraction, we strike off, in the first column, the two figures 68: and, generally, *for each figure omitted in the second column, we omit two in the first.*

It should be observed, that when places are omitted in either column, whatever would have been carried to the last figure retained, had no figures been omitted, is always to be added to that figure. Having found the figure 8 of the root, we need not annex it in the first column, nor need we annex any subsequent figures of the root, since they would all fall at the right, among the rejected figures. Hence, neither 8, nor any subsequent figures of the root, will change the available part of the first column.

In the next operation, we divide .000141586 by 1.4772, omitting the figure 8 of the divisor: this gives the figure 9 of the root. We then strike off the figures 4.0, in the first column, and multiplying by .00009, we form the next divisor in the second column, -1.4769, and the next dividend in the 3d column, .000008663. Striking off 5 in this divisor, we find the next figure of the root, which is 5.

It is now evident that the products from the first column, will fall in the second, among the rejected figures at the right; we need, therefore, in future, take no notice of them.

Omitting the right hand figure, the next divisor will be 1.476, and the next figure of the root 8. Then omitting 6 in the

divisor, we obtain the quotient figure 8: omitting 7 we obtain 6, and omitting 4 we obtain 7, the last figure to be found. We have thus found the root $x = 1.356895876 \ldots$; and all similar examples are wrought after the same manner.

The next operation is to find the root whose initial figures are 1.6, to nine decimal places. The operations are entirely similar to those just explained.

We find for the second root, $x = 1.69202141$.

For the negative root, change the signs of the second and fourth terms (Art. 280), and we have,

<table>
<tr><td>1 — 0</td><td>— 7</td><td>— 7 (3.0489173396</td></tr>
<tr><td>3</td><td>9</td><td>+ 6</td></tr>
<tr><td>3</td><td>2</td><td>— 1</td></tr>
<tr><td>3</td><td>18</td><td>.814464</td></tr>
<tr><td>6</td><td>20</td><td>— .185536</td></tr>
<tr><td>3</td><td>.3616</td><td>.166382592</td></tr>
<tr><td>9.04</td><td>20.3616</td><td>— 19153408</td></tr>
<tr><td>4</td><td>.3632</td><td>18791228</td></tr>
<tr><td>9.08</td><td>20.7248</td><td>—362180</td></tr>
<tr><td>4</td><td>73024</td><td>208875</td></tr>
<tr><td>9.128</td><td>20.797824</td><td>— 153305</td></tr>
<tr><td>8</td><td>73088</td><td>146212</td></tr>
<tr><td>9.136</td><td>20.87091|2</td><td>—7093</td></tr>
<tr><td>8</td><td>823|0</td><td>6266</td></tr>
<tr><td>|..|9.1|44</td><td>20.87914|2</td><td>—827</td></tr>
<tr><td></td><td>823|0</td><td>626</td></tr>
<tr><td></td><td>20.8873|7</td><td>—201</td></tr>
<tr><td></td><td>|9</td><td>188</td></tr>
<tr><td></td><td>20.8874|6</td><td>—13</td></tr>
<tr><td></td><td>|9</td><td>12</td></tr>
<tr><td></td><td>2|0.|8|8|7|5</td><td>1</td></tr>
</table>

4. Find the roots of the equation

$$x^3 + 11x^2 - 102x + 181 = 0.$$

The functions are

$$\begin{aligned} X &= x^3 + 11x^2 - 102x + 181 \\ X_1 &= 3x^2 + 22x - 102 \\ X_2 &= 122x - 393 \\ X_3 &= + ; \end{aligned}$$

and the signs of the leading terms are all $+$; hence, the substitution of $-\infty$ and $+\infty$ must give three real roots.

To discover the situation of the roots, we make the substitutions

$x = 0$	which gives	$+ - - +$	two variations
$x = 1$	"	$+ - - +$	"
$x = 2$	"	$+ - - +$	"
$x = 3$	"	$+ - - +$	"
$x = 4$	"	$+ + + +$	no variation;

hence the two positive roots are between 3 and 4, and we must therefore transform the several functions into others, in which x shall be diminished by 3. Thus we have (Art. 314),

$$\begin{aligned} Y &= y^3 + 20y^2 - 9y + 1 \\ Y_1 &= 3y^2 + 40y - 9 \\ Y_2 &= 122y - 27 \\ Y_3 &= + \end{aligned}$$

Make the following substitutions in these functions, viz.:

$y = 0$	signs	$+ - - +$	two variations
$y = .1$	"	$+ - - +$	"
$y = .2$	"	$+ - - +$	"
$y = .3$	"	$+ + + +$	no variation;

hence, the two positive roots are between 3.2 and 3.3, and we must again transform the last functions into others, in which y shall be diminished by .2. Effecting this transformation, we have

$$\begin{aligned} Z &= z^3 + 20.6z^2 - .88z + .008 \\ Z_1 &= 3z^2 + 41.2z - .88 \\ Z_2 &= 122z - 2.6 \\ Z_3 &= +. \end{aligned}$$

Let $z = 0$	then signs are		$+ - - +$	two variations,
$z = .01$	"	"	$+ - - +$	"
$z = .02$	"	"	$- - - +$	one variation,
$z = .03$	"	"	$+ + + +$	no variation;

hence we have 3.21 and 3.22 for the positive roots, and the sum of the roots is -11; therefore, $-11 - 3.21 - 3.22 = -17.4$, is the negative root, nearly.

For the positive root, whose initial figures are 3.21, we have

$$x = 3.21312775;$$

and for the root whose initial figures are 3.22, we have

$$x = 3.229522121;$$

and for the negative root,

$$x = -17.44264896.$$

EXAMPLES.

1. Find a root of the equation $x^3 + x^2 + x - 100 = 0$.
Ans. 4.2644299731.

2. Find the roots of the equation $x^4 - 12x^2 + 12x - 3 = 0$.

$$Ans. \begin{cases} +2.858083308163 \\ +\ \ .606018306917 \\ +\ \ .443276939605 \\ -3.907378554685. \end{cases}$$

3. Find the roots of the equation $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$.

$$Ans. \begin{cases} +5.2360679775 \\ +\ \ .7639320225 \\ +2.7320508075 \\ -\ \ .7320508075. \end{cases}$$

4. Find the roots of the equation

$$x^5 - 10x^3 + 6x + 1 = 0.$$

$$Ans. \begin{cases} -3.0653157912983 \\ -\ \ .6915762804900 \\ -\ \ .1756747992883 \\ +\ \ .8795087084144 \\ +3.0530581626622. \end{cases}$$

$\frac{xy}{z} = a$ $\frac{xz}{y} = b$ $\frac{yz}{x} = c$

$xy = az$ $xz = by$ $x = \frac{by}{z}$

$yz = cx$

$y = \frac{xz}{b}$

$y = \frac{cx}{z}$

$\frac{xz}{b} = \frac{cx}{z}$

$\frac{az}{y} = \frac{by}{z}$

$az^2 = by^2$

$x^2y^2 = a^2z^2$

$x^2y^2 = a^2by^2$

$x^2 = ab$

$x = \sqrt{ab}$

5 Quen. $xz^2 = bcx$

18 Page $z^2\sqrt{ab} = bc\sqrt{ab}$

$z^2 = bc$

$z = \sqrt{bc}$

$y\sqrt{ab} = a\sqrt{bc}$

$y\sqrt{ab} = \sqrt{abac}$

$y = \sqrt{ac}$

When this you see remember me
Though many miles [illegible]
[illegible]